MAT 545: Complex Geometry
Fall 2008

Notes on Lefschetz Decomposition

1 Statement

Let (M, J, ω) be a Kahler manifold. Since ω is a closed 2-form, it induces a well-defined homomorphism

$$L\colon H^k(M) \longrightarrow H^{k+2}(M), \qquad L\big([\alpha]\big) = [\omega \wedge \alpha].$$

Hard Lefschetz Theorem: If (M, J, ω) is a compact Kahler m-manifold, the homomorphism

$$L^r : H^{m-r}(M) \longrightarrow H^{m+r}(M) \tag{1}$$

is an isomorphism for all $r \geq 0$.

Since ω is a $(1,1)$-form, by the Hodge decomposition theorem the above claim is equivalent to the statement that each of the homomorphisms

$$L^{m-(p+q)}\colon H^{p,q}(M) \longrightarrow H^{m-q,m-p}(M)$$

with $p+q \leq m$ is an isomorphism. In the Hodge diamond, L corresponds to moving up 1 step along the vertical lines; the above isomorphisms take the (p,q)-slot to its reflection about the horizontal diagonal. Thus, the Hodge diamond of a compact Kahler manifold is symmetric about the horizontal diagonal. This fact also follows from the Hodge theorem (which implies that the Hodge diamond is symmetric about the vertical diagonal) and the Kodaira-Serre duality (symmetry about the center of the diamond). Furthermore, the Hard Lefschetz Theorem provides a relative restriction on the numbers along each of the vertical lines in the Hodge diamond: these numbers are non-decreasing in the bottom half and non-increasing in the top half (see (2) below).

For each $r = 0, 1, \ldots, m$, let

$$PH^{m-r}(M) \equiv \big\{\alpha \in H^r(M)\colon\ L^{r+1}\alpha = 0\big\}$$

be the primitive cohomology of M. By the Hard Lefschetz Theorem, $r+1$ is the smallest value of s such that the homomorphism

$$L^s\colon H^{m-r}(M) \longrightarrow M^{m-r+2s}(M)$$

may have a kernel; furthermore,

$$H^{m-r}(M) = PH^{m-r}(M) \oplus L\big(H^{m-r-2}(M)\big).$$

Thus, the Hard Lefschetz Theorem implies that

$$H^k(M) = \bigoplus_{\substack{s\ge 0\\ k-2s\le m}} L^s\big(PH^{k-2s}(M)\big). \tag{2}$$

On the other hand, (2) implies (1) immediately.

Let $H\in H_2(M)$ be the Poincare dual of $[\omega]$ and denote by

$$\cdot_M\colon H_k(M)\otimes H_l(M) \longrightarrow H_{k+l-2m}(M)$$

the homology intersection product on M (the Poincare dual of the cup product). Via Poincare Duality, the Hard Lefschetz Theorem is equivalent to the statement that

$$(H\cdot_M)^r\colon H_{m+r}(M) \longrightarrow H_{m-r}(M), \qquad \eta \longrightarrow \underbrace{H\cdot_M \ldots H\cdot_M}_{r}\eta,$$

is an isomorphism for all $r\ge 0$. Furthermore,

$$\begin{aligned} PD_M\big(L^r(PH^{m-r}(M))\big) &= PD_M\big(\{\alpha\in H^{m+r}(M)\colon\ L(\alpha)=0\}\big)\\ &= \big\{\eta\in M_{m-r}(M)\colon H\cdot_M \eta=0\big\}. \end{aligned}$$

Thus, the primitive k-th cohomology of M, with $k\le m$, corresponds to the k-cycles that are "disjoint" from H; thus, it is the image of the homomorphism

$$H_k(M-V) \longrightarrow H_k(M)$$

induced by inclusion.

The Poincare dual of the Fubini-Study symplectic form ω_{FS} on $\mathbb{P}^n$ is the hyperplane class $H\approx\mathbb{P}^{n-1}$, since

$$\int_{\mathbb{P}^1}\omega_{FS} = 1 = \mathbb{P}^1\cdot_{\mathbb{P}^n}\mathbb{P}^{n-1}$$

and $H^2(\mathbb{P}^n)$ is one-dimensional. If $M\subset\mathbb{P}^n$ is a compact Kahler submanifold of dimension m,

$$PD_M\big(\omega_{FS}|_M\big) = H\cap M.$$

Thus, the Hard Lefschetz Theorem in this case is equivalent to the statement that

$$\mathbb{P}^{n-r}\cap\colon H_{m+r}(M) \longrightarrow H_{m-r}(M)$$

is an isomorphism. The primitive k-th cohomology of M, with $k\le m$, corresponds to the k-cycles in M that are "disjoint" from H. Thus, they lie in

$$M-H\subset\mathbb{P}^n-H=\mathbb{C}^n\,.$$

2 Proof

The Hard Lefschetz Theorem is a consequence of Hodge identities and the Hodge theorem. Let (M,J,ω) be a Kahler m-manifold and

$$\Lambda\colon A^k(M)\longrightarrow A^{k-2}(M)$$

the adjoint of the homomorphism $L=\omega\wedge$ with respect to the inner-product induced by (ω,J). Denote by

$$\Delta\equiv\Delta_d\equiv dd^*+d^*d\colon A^k(M)\longrightarrow A^k(M)$$

be the corresponding d-Laplacian and let

$$\mathcal{H}^p\equiv\big\{\alpha\in A^p(M)\colon\ \Delta\alpha=0\big\}.$$

Hodge Identities: If (M,J,ω) is a Kahler m-manifold

$$L\Delta=\Delta L,\qquad \Lambda\Delta=\Delta\Lambda\,, \tag{3}$$
$$L\Lambda-\Lambda L=(m-k)\,\mathrm{Id}\colon A^k(M)\longrightarrow A^k(M). \tag{4}$$

Hodge Theorem: If (M,J,ω) is a compact Kahler m-manifold, the homomorphism

$$\mathcal{H}^k\longrightarrow H^k(M),\qquad \alpha\longrightarrow[\alpha],$$

is well-defined and is an isomorphism.

The second statement is valid for any Riemannian manifold, while (4) is a point-wise statement and thus follows from a direct check for $\mathbb{C}^n$. The identities (3) imply that L and Λ restrict to homomorphisms

$$L\colon\mathcal{H}^k\longrightarrow\mathcal{H}^{k+2},\qquad \Lambda\colon\mathcal{H}^k\longrightarrow\mathcal{H}^{k-2}\,.$$

By Hodge theorem, it is sufficient to prove the analogue of the Hard Lefschetz Theorem for $\mathcal{H}^*$. From (4), we obtain the following lemma.

Lemma 1 *If $\alpha\in\mathcal{H}^k$, then for all $s\ge 1$*

$$\Lambda L^s\alpha=L^{s-1}\big(C_{k,s}\alpha+L\Lambda\alpha\big)$$

for some $C_{k,s}\in\mathbb{Z}$ such that $C_{k,s}=0$ if and only if $s=m-k+1$.

Corollary 2 *If $k\le m$, $\alpha\in\mathcal{H}^k$, and $L^{m-k+1}\alpha=0$, then*
(a) $\Lambda\alpha=0$;
(b) $L^s\alpha\neq0$ for all $s=0,1,\dots,m-k$ if $\alpha\neq0$.

Proof: Both statements hold for $k<0$. Suppose $0 \le k \le m$ and both statements are valid for all $k'<k$. By Lemma 1,

$$0 = \Lambda L^{m-k+1}\alpha = L^{m-k}\big(C_{k,m-k+1}\alpha + L\Lambda\alpha\big) = L^{m-k+1}(\Lambda\alpha).$$

Thus, $\Lambda\alpha=0$ by the $k-2$, $s=m-k+1$ case of (b). The $s=0$ case of (ii) clearly holds. Suppose $1\le s\le m-k$ and (ii) holds for all $s'<s$. By Lemma 1,

$$\Lambda L^s\alpha = L^s(\Lambda\alpha) + C_{k,s}L^{s-1}\alpha = C_{k,s}L^{s-1}\alpha \neq 0 \qquad \text{if} \quad \alpha \neq 0,$$

since $\Lambda\alpha=0$ by (a) and $C_{k,s}\neq 0$.

Corollary 3 *For all* $k\le m$,

$$\mathcal{H}^k = \ker L^{m-k+1}|_{\mathcal{H}^k} \oplus L\mathcal{H}^{k-2}\,, \qquad \mathcal{H}^{2m-k} = L^{m-k}\mathcal{H}^k\,.$$

Proof: (1) Suppose $\alpha\in\mathcal{H}^k$; then $L^{m-k+s}\alpha=0$ for some $s\ge 1$. If $s=1$, there is nothing to prove. Suppose $s\ge 2$ and

$$\ker\big(L^{m-k+s-1}\colon \mathcal{H}^k \longrightarrow \mathcal{H}^{2m-k+2s-2}\big) \subset \ker L^{m-k+1}|_{\mathcal{H}^k} \oplus L\mathcal{H}^{k-2}\,.$$

By Lemma 1,

$$0 = \Lambda L^{m-k+s}\alpha = L^{m-k+s-1}\big(C_{k,m-k+s}\alpha + L\Lambda\alpha\big).$$

Thus, $C_{k,m-k+s}\alpha + L\Lambda\alpha \in \ker L^{m-k+1}|_{\mathcal{H}^k} \oplus L\mathcal{H}^{k-2}$; since $C_{k,m-k+s}\neq 0$, it follows that

$$\alpha \in \ker L^{m-k+1}|_{\mathcal{H}^k} \oplus L\mathcal{H}^{k-2}.$$

(2) If $k=m$, there is nothing to prove. Suppose $k<m$ and the statement holds for all $k'>k$. If $\alpha\in\mathcal{H}^{2m-k}$, then $L^s\alpha=0$ for some $s\ge 0$. If $s=0$, $\alpha=L^{m-k}0$. Suppose $s\ge 1$ and

$$\ker\big(L^{s-1}\colon \mathcal{H}^{2m-k} \longrightarrow \mathcal{H}^{2m-k+2s-2}\big) \subset L^{m-k}\mathcal{H}^k\,.$$

By Lemma 1,

$$0 = \Lambda L^s\alpha = L^{s-1}\big(C_{2m-k,s}\alpha + L\Lambda_\alpha\big).$$

Thus, $C_{2m-k,s}\alpha + L\Lambda\alpha \in L^{m-k}\mathcal{H}^k$; since $C_{2m-k,s}\neq 0$, it follows that

$$\alpha \in L^{m-k}\mathcal{H}^k + L\mathcal{H}^{2m-k-2}\,.$$

On the other hand, by the $k'=k+2$ cases of the first and second statements,

$$L\mathcal{H}^{2m-k-2} = L^{m-k}\mathcal{H}^{m-k}.$$

Corollaries 2 and 3 imply the analogue of the Hard Lefschetz Theorem for $\mathcal{H}^*$, since the homomorphism in (1) is injective by the former and surjective by the latter.

3 Applications

The Hodge theorem provides restrictions on topological and complex manifolds that admit a Kahler structure. One of the standard invariants of a topological manifold M is the k-th Betti number,

$$h_k(M) \equiv \dim_{\mathbb{R}} H^k(M;\mathbb{R}) = \dim_{\mathbb{C}} H^k(M;\mathbb{C}) = \dim_{\mathbb{R}} H^k_{deR}(M;\mathbb{R});$$

the last equality holds if M admits a smooth structure. If M is a compact $2m$-dimensional topological manifold that admits a Kahler structure, then $h_{2r+1}(M)$ is even for all $r \in \mathbb{Z}$ and $h_{2r}(M) > 0$ for all $r = 0, 1, \ldots, m$. Furthermore, $H^2(M;\mathbb{R})$ contains an element α such that $\alpha^m \neq 0$. If (M, J) is a compact complex manifold that admits a compatible Kahler structure, then

$$h^{p,q}(M) \equiv \dim_{\mathbb{C}} H^{p,q}_d(M) = \dim_{\mathbb{C}} H^{p,q}_{\bar\partial}(M) \quad \forall\, p, q,$$
$$h^{p,q}(M) = h^{q,p}(M) \quad \forall\, p, q, \quad h^k(M) = \sum_{p+q=r} h^{p,q}(M) \quad \forall\, r.$$

Furthermore, the homology class of any analytic subvariety in (M, J) is non-zero in the homology of M, as is every holomorphic p-form in $H^p(M)$.

The Hard Lefschetz Theorem provides additional restrictions. If M is a compact $2m$-dimensional topological manifold that admits a Kahler structure, then $H^2(M;\mathbb{R})$ contains an element α such that the homomorphisms

$$H^{m-r}(M) \longrightarrow H^{m+r}(M), \quad \beta \longrightarrow \alpha^r \wedge \beta, \qquad r \geq 0,$$

are isomorphisms. If (M, J) is a compact complex m-manifold that admits a compatible Kahler structure, then $H^{1,1}_d(M) \cap H^2(M;\mathbb{R})$ contains an element α such that the homomorphisms

$$H^{p,q}_d(M) \longrightarrow H^{m-q,m-p}_d(M), \quad \beta \longrightarrow \alpha^{m-p-q} \wedge \beta, \qquad p+q \leq m,$$

are isomorphisms.

If M is a compact topological oriented $4k$-dimensional manifold, the pairing

$$Q\colon H^{2k}(M;\mathbb{R}) \otimes H^{2k}(M;\mathbb{R}) \longrightarrow \mathbb{R}, \qquad \alpha \otimes \beta \longrightarrow \big\langle \alpha \cup \beta, [M] \big\rangle .$$

is non-degenerate by Poincare duality and is symmetric. Thus, $H^{2k}(M;\mathbb{R})$ admits a basis with respect to which this pairing is diagonal with each of the non-zero entries equal to $+1$ or -1. Let $\lambda_\pm(M) = \lambda_\pm(Q)$ denote the number of ± 1 entries; this number is determined by the bilinear form Q and thus by the topology and the orientation of M. So, is the number

$$\sigma(M) = \lambda_+(M) - \lambda_-(M),$$

which is known as the signature of M. If in addition J is a complex structure on M, Q restricts to a non-degenerate symmetric pairing $Q^{k,k}$ on $H^{k,k}(M) \cap H^{2k}(M;\mathbb{R})$. Let $\lambda^{p,p}_\pm(M) = \lambda_\pm(Q^{p,p})$.

Index Theorem for Surfaces: If (M,J) is a compact connected complex surface ($\dim_{\mathbb{C}} M = 2$) that admits a compatible Kahler structure, then $\lambda_+^{1,1}(M) = 1$.

Proof: Let ω be a symplectic form on M compatible with J and $\mathcal{H}^{1,1}$ the corresponding space of harmonic $(1,1)$-forms. By the Hard Lefschetz Theorem,

$$\mathcal{H}^{1,1} = \mathbb{C}\omega \oplus \big\{\alpha \in \mathcal{H}^{1,1} \colon \omega \wedge \alpha = 0\big\} \equiv V_+ \oplus V_0.$$

This decomposition is Q-orthogonal, and Q restricted to $V_+ \cap H^2(M;\mathbb{R})$ is positive-definite. Thus, it is sufficient to show that

$$Q(\alpha,\alpha) \le 0 \qquad \forall\, \alpha \in V_0 \cap H^2(M;\mathbb{R}).$$

Given $p \in M$, let (z_1, z_2) be holomorphic coordinates around p on M so that

$$\omega|_p = -\frac{1}{2}\mathrm{Im}\big(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2\big)|_p = dx_1 \wedge dy_1 + dx_2 \wedge dy_2\,,$$

where $z_j = x_j + \mathfrak{i} y_j$. If $\alpha \in \mathcal{H}^{1,1} \cap H^2(M;\mathbb{R})$,

$$\alpha|_p = \big(A\, dx_1 \wedge dy_1 + B\, dx_2 \wedge dy_2 + C(dx_1 \wedge dx_2 + dy_1 \wedge dy_2) + D(dx_1 \wedge dy_2 - dy_1 \wedge dx_2)\big)|_p$$

for some $A, B, C, D \in \mathbb{R}$. If in addition $\omega \wedge \alpha = 0$, $A = -B$ and

$$\alpha|_p \wedge \alpha|_p = -2(A^2 + C^2 + D^2) dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2\big|_p.$$

Thus, $\alpha|_p \wedge \alpha|_p$ is a non-positive multiple of the volume form on M for all $p \in M$ and thus $Q(\alpha,\alpha) \le 0$ as needed.

If (M,J) is a compact complex $2k$-manifold that admits a compatible Kahler structure, then

$$\sigma(M) = \sum_{p+q \equiv 0 (\mathrm{mod}\, 2)} h^{p,q}(M).$$

This is deduced from the *Hodge-Riemann bilinear relations* (a single formula) in Griffiths&Harris, pp123-126. As explained in the top half of p124, these relations follow from the Hard Lefschetz Theorem and quite a bit of representation theory (a classical subject).

There is an important typo on p123, in the line before the statement of the Hodge-Riemann bilinear relations: "$k = p + q$" should in fact be "$n - k = p + q$".