MAT 545: Complex Geometry
Problem Set 5 Solutions

Problem 1 (10 pts)

Suppose (M,J) is an almost complex manifold, g is a J-compatible Riemannian metric on M, and ∇ is the Levi-Civita connection of g (thus, J is a complex structure in the fibers of the vector bundle $TM\longrightarrow M$ which preserves g; ∇ is g-compatible and $[X,Y]=\nabla_X Y-\nabla_Y X$ for any two vector fields X,Y on M). Show that $\nabla J=0$ if and only if (M,J,g) is Kahler (you can use either of the equivalent conditions in PS2,#1 as the integrability criterion for J).

The nondegenerate 2-form determined by (g,J) is given by

$$\omega(X,Y)=g(JX,Y).$$

Given a fixed point $p\in M$, let X,Y,Z be vector fields on M such that

$$\begin{aligned} &\nabla X|_p=0, \quad \nabla Y|_p=0, \quad \nabla Z|_p=0;\\ \Longrightarrow \qquad &\nabla(JX)\big|_p=(\nabla J)_p X_p, \quad \nabla(JY)\big|_p=(\nabla J)_p Y_p \quad \nabla(JZ)\big|_p=(\nabla J)_p Z_p,\\ &[X,Y]_p=0, \quad [X,Z]_p=0, \quad [Y,Z]_p=0. \end{aligned}$$

(i) Suppose $\nabla J=0$. We first show that the Nijenhuis tensor of J,

$$N_J(X,Y)\equiv\frac{1}{2}\Big([X,Y]+J[X,JY]+J[JX,Y]-[JX,JY]\Big)$$

vanishes. With X and Y as above,

$$N_J(X,Y)_p=\big(J(\nabla_X J)Y-J(\nabla_Y J)X-(\nabla_{JX}J)Y+(\nabla_{JY}J)X\big)_p=0,$$

since $\nabla J=0$; thus, (M,J) is a complex manifold. We next show that $d\omega=0$. With X,Y,Z as above

$$\begin{aligned} &\{d\omega\}_p(X,Y,Z)\\ &\quad=\big(X\omega(Y,Z)+Y\omega(Z,X)+Z\omega(X,Y)-\omega([X,Y],Z)-\omega([Y,Z],X)-\omega([Z,X],Y)\big)_p\\ &\quad=\big(Xg(JY,Z)+Yg(JZ,X)+Zg(JX,Y)\big)_p\\ &\quad=\big(g((\nabla_X J)Y,Z)+g((\nabla_Y J)Z,X)+g((\nabla_Z J)X,Y)\big)_p=0, \end{aligned}$$

since $\nabla J=0$. Thus, (M,ω) is a symplectic manifold, and (M,g,J) is Kahler.

(ii) Suppose (M,g,J) is Kahler; we show that $\nabla J=0$.

Solution (a): By Lemma in G&H, p107, there is a complex coordinate system $z=(z_1,\ldots,z_m)$, with $z_k=x_k+\mathrm{i}y_k$, centered around $p\in M$ such that

$$g=\sum_k\big(dx_k\otimes dx_k+dy_k\otimes dy_k\big)+O(z^2).$$

Since ∇ is determined by the values of g and of its first derivatives, $(\nabla T)_p$ agrees with $(DT)_p$ for every tensor T, where D is the connection for the flat metric. In particular, $(\nabla J)_p = 0$.

Solution (b): From the proof in part (i),

$$\begin{aligned} N_J(X,Y)_p &= \big(J(\nabla_X J)Y - J(\nabla_Y J)X - (\nabla_{JX} J)Y + (\nabla_{JY} J)X\big)_p; \\ \{d\omega\}_p(X,Y,Z) &= \big(g((\nabla_X J)Y,Z) + g((\nabla_Y J)Z,X) + g((\nabla_Z J)X,Y)\big)_p. \end{aligned}$$

Since $J^2 = -\mathrm{I}d$, $g(J\cdot,\cdot) = -g(\cdot,J\cdot)$, and ∇ is g-compatible,

$$J(\nabla J) = -(\nabla J)J, \qquad g\big((\nabla_X J)Y,Z\big) = -g\big(Y,(\nabla_X J)Z\big).$$

This all gives

$$2g\big((\nabla_Z J)X, JY\big) = g\big(N_J(X,Y),Z\big) + d\omega(JX,Y,Z) + d\omega(X,JY,Z).$$

Since M is Kahler, $N_J, d\omega = 0$ and thus ∇J.

Solution (c): Let D be the metric connection in the holomorphic tangent $(TM,J) \longrightarrow M$ with the hermitian inner-product h by g; in particular, $DJ = 0$. It is sufficient to show that $D = \nabla$, i.e. D is g-compatible and torsion-free. The former is immediate, since D is h-compatible. Since M is Kahler, the "torsion" τ of D defined by (*) in G&H, p76, is zero (G&H, p107). It is thus sufficient to check that

$$\tau(X,Y) = D_X Y - D_Y X - [X,Y]$$

for any two vector fields X, Y on M. Let $v_1, \ldots, v_m$ be a local h-orthonormal $\mathbb{C}$-frame for (TM,J) and $\varphi_1, \ldots, \varphi_m$ the dual frame for $(T^*M, J) = (T^*M^{1,0}, \mathfrak{i})$. Then, for each i

$$d\varphi_i = \sum_j \psi_{ij} \wedge \varphi_j + \tau_i$$

for unique 1-forms ψ_{ij} and $(2,0)$-form τ_i such that $\psi_{ji} = -\bar{\psi}_{ij}$. Furthermore,

$$Dv_i = -\sum_k \psi_{ki} \otimes v_k$$

for all i; see p77. Thus,

$$\begin{aligned} \varphi_k\big(D_{v_i} v_j - D_{v_j} v_i - [v_i, v_j]\big) &= -\psi_{kj}(v_i) + \psi_{ki}(v_j) + \{d\varphi_k\}(v_i, v_j) \\ &= -\psi_{kj}(v_i) + \psi_{ki}(v_j) + \big(\psi_{kj}(v_i) - \psi_{ki}(v_j) + \tau_k(v_i, v_j)\big) = \tau_k(v_i, v_j). \end{aligned}$$

Thus, τ_k is indeed the v_k-component of the usual torsion and its vanishing implies that D is the Levi-Civita connection for (M, g, J).

Problem 3 (5 pts)

Let $M_n=(\mathbb{C}^n-0)/\sim$, where $z\sim 2^k z$ for every $k\in\mathbb{Z}$.
(a) Show that M_n is a complex manifold (with the complex structure inherited from $\mathbb{C}^n$). What simple smooth manifold is M_n diffeomorphic to?
(b) Give at least two reasons why M_n does not admit a Kahler metric for $n\ge 2$.

(a) Let S^{2n-1} denote the unit sphere in $\mathbb{C}^n$. The diffeomorphism

$$f: \mathbb{R}\times S^{2n-1} \longrightarrow \mathbb{C}^n-0, \qquad (s,w)\longrightarrow 2^s w,$$

is $\mathbb{Z}$-equivariant with respect to the above action on $\mathbb{C}^n-0$ and the action

$$k\cdot(s,z) = (s+k,z)$$

on $\mathbb{R}\times S^{2n-1}$. Thus, M_n is diffeomorphic to $(\mathbb{R}/\mathbb{Z})\times S^{2n-1}$. Since the standard action of $\mathbb{Z}$ on $\mathbb{R}$ is properly discontinuous, so is the action of $\mathbb{Z}$ on $\mathbb{C}^n-0$. Thus, the quotient map $q:\mathbb{C}^n-0\longrightarrow M_n$ is a covering projection. Since $\mathbb{Z}$ acts by holomorphic transformations on $\mathbb{C}^n-0$, i.e. the complex structure on $\mathbb{C}^n-0$ is preserved by the $\mathbb{Z}$-action, the complex structure on $\mathbb{C}^n-0$ descends to a complex structure on the quotient M_n.

(b) If $n\ge 2$, by the Kunneth formula

$$H^1_{deR}(M_n;\mathbb{R})\approx\mathbb{R}, \qquad H^2_{deR}(M_n;\mathbb{R})=\{0\}, \qquad H_2(M_n;\mathbb{Z})=\{0\}.$$

However, the odd betti numbers of a compact Kahler manifolds are even, while the second betti number is non-zero; this contradicts the first statements above. The third statement implies that the complex one-dimensional tori $q(L-0)\subset M_n$, where $L\subset M_n$ is a one-dimensional linear subspace, are trivial in the homology of M; this is yet another reason that M_n is not Kahler.

Problem 4 (10 pts)

Let $M=\mathbb{R}^4/\sim$, where

$$(s,t,x,y)\sim\big(s+k,t+l,x+m,y+lx+n\big) \qquad \forall\ (s,t,x,y)\in\mathbb{R}^4,\ (k,l,m,n)\in\mathbb{Z}^4\,.$$

Show that
(a) this is an equivalence relation;
(b) M is a compact symplectic manifold (with the symplectic form, i.e. closed non-degenerate 2-form, inherited from the standard symplectic form on $\mathbb{R}^4$, i.e. $ds\wedge dt+dx\wedge dy$).
(c) M does not admit an integrable complex structure compatible with this symplectic form.
Note 1: this is the first known example (due to W. Thurston'76) of a symplectic manifold that admits no Kahler structure.
Note 2: in contrast, every symplectic manifold (M,ω) admits an almost complex structure compatible with ω; the space of ω-compatible almost complex structures is contractible.

For each $(k,l,m,n)\in\mathbb{Z}^4$, define

$$\varphi_{(k,l,m,n)}\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^4 \qquad \text{by} \qquad \varphi_{(k,l,m,n)}(s,t,x,y) = \big(s+k, t+l, x+m, y+lx+n\big).$$

Then, $(s,t,x,y) \sim (s',t',x',y')$ if and only if $(s',t',x',y') = \varphi_{(k,l,m,n)}(s,t,x,y)$ for some $(k,l,m,n)\in\mathbb{Z}^4$.

(a) Reflexivity: $(s,t,x,y)=\varphi_{(0,0,0,0)}(s,t,x,y)$.
Symmetry: if $(s',t',x',y')=\varphi_{(k,l,m,n)}(s,t,x,y)$, then

$$(s,t,x,y)=\varphi_{(-k,-l,-m,-n+lm)}(s',t',x',y');$$

Transitivity: if $(s_i,t_i,x_i,y_i)=\varphi_{(k_i,l_i,m_i,n_i)}(s_{i-1},t_{i-1},x_{i-1},y_{i-1})$ for $i=0,1$, then

$$(s_2,t_2,x_2,y_2)=\varphi_{(k_1+k_2,l_1+l_2,m_1+m_2,n_1+n_2+m_1l_2)}(s_0,t_0,x_0,y_0).$$

It follows that M is the quotient of $\mathbb{R}$ by the group $G=\mathbb{Z}^4$ with the composition law

$$(k_1,l_1,m_1,n_1)\cdot(k_2,l_2,m_2,n_2)=(k_1+k_2,l_1+l_2,m_1+m_2,n_1+n_2+m_1l_2),$$

which is acting on the right.

(b) For every $(s,t,x,y)\in\mathbb{R}^4$, there exists $(k,l,m,n)\in\mathbb{Z}^4$ such that

$$\varphi_{(k,l,m,n)}(s,t,x,y)\in[0,1]^4\,.$$

Since $[0,1]^4$ is compact, it follows that so is M. The action group G on $\mathbb{R}^4$ is properly discontionuos: if p is any point and $B_p(1/2)$ is the open ball of radius $1/2$ around p, in the square or round metric, then

$$B_p(1/2)\cap\varphi_{(k,l,m,n)}\big(B_p(1/2)\big)=\emptyset \qquad \forall\,(k,l,m,n)\in\mathbb{Z}^4-0.$$

Thus, the quotient map $q\colon\mathbb{R}^4\longrightarrow M$ is a covering projection. Since G acts on $\mathbb{R}^4$ by diffeomorphisms, the smooth structure on $\mathbb{R}^4$, descends to M. Since

$$\varphi^*_{k,l,m,n}(ds\wedge dt+dx\wedge dy)=ds\wedge dt+dx\wedge(dy+ldx)=ds\wedge dt+dx\wedge dy,$$

the G acts on $\mathbb{R}^4$ preserves the symplectic form, which thus descends to a symplectic on M.

(c) Since $\mathbb{R}^4\longrightarrow M$ is a covering projection, $\pi_1(M)=G$ and $H_1(M;\mathbb{Z})=G/[G,G]$. Since the projection on the first three coordinates $G\longrightarrow\mathbb{Z}^3$ is a group homomorphism, $[G,G]\subset 0^3\times\mathbb{Z}$. Since G is not abelian, it follows that

$$H_1(M;\mathbb{R})=H_1(M;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}\approx\mathbb{R}^3;$$

(in fact $[G,G]=0^3\times\mathbb{Z}$, but this is not needed). Thus, the first betti number of M is odd and therefore M does not admit a Kahler metric.

Problem 5 (5 pts)

Let (X, J, g) be a Kahler manifold and ω its symplectic form. Show that ω is harmonic with respect to g.

Let L be the wedging with ω operator and Δ either of the three laplacians. By one of the main Hodge relations, $\Delta L = L\Delta$ (this is used in the proof of the Lefschetz theorem). Thus,

$$\Delta\omega = \Delta(L1) = L(\Delta 1) = L0 = 0.$$

Problem 6 (10 pts)

Let M be a compact complex manifold that admits a Kahler metric.
(a) Let α be a (p,q)-form on M such that $d\alpha = 0$. Show that the following are equivalent:

(i) $\alpha = d\beta$ for some $(p+q-1)$-form β;

(ii) $\alpha = \partial\beta$ for some $(p-1, q)$-form β;

(iii) $\alpha = \bar\partial\beta$ for some $(p, q-1)$-form β;

(iv) $\alpha = \partial\bar\partial\beta$ for some $(p-1, q-1)$-form β.

(b) Let ω and ω' be symplectic forms compatible with the complex structure on M (thus ω and ω' arise from Kahler metrics on M). If $[\omega] = [\omega'] \in H^2_{deR}(M)$, show that $\omega' = \omega + \mathfrak{i}\partial\bar\partial f$ for some $f \in C^\infty(M;\mathbb{R})$.

(a) (iv) $\Longrightarrow$ (i),(ii),(iii), since $d = \partial + \bar\partial$, $\partial^2 = 0$, $\bar\partial^2 = 0$, and $\partial\bar\partial = -\bar\partial\partial$ on any complex manifold.
(i) $\Longrightarrow$ (ii),(iii): Since $\alpha = d\beta$, α is orthogonal to the harmonic forms (since M is Kahler, harmonic forms for d, ∂, and $\bar\partial$ are the same). Thus, by the Hodge decomposition for ∂ and $\bar\partial$,

$$\alpha = \partial\gamma_- + \partial^*\gamma_+ = \bar\partial\gamma'_- + \bar\partial^*\gamma'_+ \,.$$

Since $\partial\alpha = 0$ and $\bar\partial\alpha = 0$ (because α is of pure bi-degree), $\partial^*\gamma_+, \bar\partial^*\gamma'_+ = 0$.
(ii) $\Longrightarrow$ (iv): By Hodge decomposition, $\beta = \beta_0 + \bar\partial\beta_- + \bar\partial^*\beta_+$, where β_0 is harmonic. Thus,

$$\alpha = \partial\beta = \partial\bar\partial\beta_- + \partial\bar\partial^*\beta_+ \qquad \Longrightarrow \qquad 0 = \bar\partial\partial\beta = \bar\partial\partial\bar\partial\beta_- + \bar\partial\partial\bar\partial^*\beta_+ \,.$$

Since $\bar\partial\partial = -\partial\bar\partial$ and $\bar\partial^2 = 0$ on all complex manifolds, while $\partial\bar\partial^* = -\bar\partial^*\partial$ on Kahler manifolds, it follows that

$$\bar\partial\bar\partial^*\partial\beta_+ = 0 \quad \Longrightarrow \quad \bar\partial^*\partial\beta_+ = 0 \quad \Longrightarrow \quad \partial\bar\partial^*\beta_+ = 0 \quad \Longrightarrow \quad \alpha = \partial\bar\partial\beta_- \,,$$

as required.

(b) Let $\alpha = \omega - \omega'$. By our assumptions, $\alpha \in A^{1,1}(M)$, $d\alpha = 0$, and $\alpha = d\beta$. Thus, $\alpha = \mathfrak{i}\partial\bar\partial g$ for some $g \in C^\infty(M;\mathbb{C})$ by part (a). Since α is a real $(1,1)$-form, it follows that

$$\alpha = \bar\alpha = (-\mathfrak{i})\bar\partial\partial\bar g = \mathfrak{i}\partial\bar\partial\bar g \qquad \Longrightarrow \alpha = \mathfrak{i}\partial\bar\partial(g + \bar g)/2;$$

so we can take $f = \operatorname{Re} g$.