Appendices

A Čech cohomology

This appendix contains a detailed review of Čech cohomology, including for sheaves of non-abelian groups, describes its connections with singular cohomology and principal bundles, and classifies oriented vector bundles over bordered surfaces. We carefully specify the assumptions required in each statement.

We generally follow the perspective of [32, Chapter 5]. In particular, a sheaf $\mathcal{S}$ over a topological space Y is a topological space along with a projection map $\pi:\mathcal{S}\longrightarrow Y$ so that π is a local homeomorphism. For a sheaf $\mathcal{S}$ of modules over a ring R as in [32] and in Section A.1 below, $\mathcal{S}_y\equiv\pi^{-1}(y)$ is a module over R for every $y\in Y$ and the module operations are continuous with respect to the topology of $\mathcal{S}$. For a sheaf $\mathcal{S}$ of groups (not necessarily abelian), as in Sections A.2-A.4 below, $\mathcal{S}_y$ is a group for every $y\in Y$ and the group operations are continuous with respect to the topology of $\mathcal{S}$. For a collection $\{U_\alpha\}_{\alpha\in\mathscr{A}}$ of subsets of a space Y and $\alpha_0,\alpha_1,\ldots,\alpha_p\in\mathscr{A}$, we set

$$U_{\alpha_0\alpha_1\ldots\alpha_p}=U_{\alpha_0}\cap U_{\alpha_1}\cap\ldots\cap U_{\alpha_p}\subset Y\,.$$

A.1 Identification with singular cohomology

For a sufficiently nice topological space Y and a module M over a ring R, the Čech cohomology group $\check{H}^p(Y;M)$ of Y with coefficients in the sheaf $Y\times M$ of germs of locally constant functions on Y with values in M is well-known to be canonically isomorphic to the singular cohomology group $H^p(Y;M)$ of Y with coefficients in M. Proposition A.1 below makes this precise in the $M=\mathbb{Z}_2$ case relevant to our purposes, making use of the locally H^k-simple notion of Definition 3.2. The statement and proof of this proposition apply to an arbitrary module M over a ring R. The $p=1$ case of the isomorphism of Proposition A.1 is described explicitly at the end of this section.

Proposition A.1. *Let $k\in\mathbb{Z}^{\geqslant 0}$. For every paracompact locally H^k-simple space Y, there exist canonical isomorphisms*

$$\Phi_Y: H^p(Y;\mathbb{Z}_2)\xrightarrow{\approx}\check{H}^p(Y;\mathbb{Z}_2),\qquad p=0,1\ldots,k\,. \tag{A.1}$$

If Y is another paracompact locally H^k-simple space and $f:Y\longrightarrow Y'$ is a continuous map, then the diagram

$$\begin{array}{ccc} H^p(Y';\mathbb{Z}_2) & \xrightarrow{\Phi_{Y'}} & \check{H}^p(Y';\mathbb{Z}_2) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ H^p(Y;\mathbb{Z}_2) & \xrightarrow{\Phi_Y} & \check{H}^p(Y;\mathbb{Z}_2) \end{array} \tag{A.2}$$

commutes for every $p \leqslant k$.

Proof. Let $p \in \mathbb{Z}^{\geqslant 0}$ and Y be a topological space. Denote by $\mathcal{S}_Y^p \longrightarrow Y$ the sheaf of germs of $\mathbb{Z}_2$-valued singular p-cochains on Y as in [32, 5.31], by

$$\mathrm{d}_p\colon \mathcal{S}_Y^p \longrightarrow \mathcal{S}_Y^{p+1} \tag{A.3}$$

the homomorphism induced by the usual differential in the singular cohomology theory, and by

$$\mathrm{d}_{p;Y}\colon \Gamma\big(Y;\mathcal{S}_Y^p\big) \longrightarrow \Gamma\big(Y;\mathcal{S}_Y^{p+1}\big)$$

the resulting homomorphism between the spaces of global sections. Let $\mathscr{Z}_Y^p \subset \mathcal{S}_Y^p$ be the kernel of the sheaf homomorphism (A.3) so that

$$\{0\} \longrightarrow \mathscr{Z}_Y^p \longrightarrow \mathcal{S}_Y^p \xrightarrow{d_p} \mathscr{Z}_Y^{p+1} \tag{A.4}$$

is an exact sequence of sheaves. If Y is locally path-connected, $\mathscr{Z}_Y^0 = Y \times \mathbb{Z}_2$.

From now on, we assume that Y is paracompact. By the exactness of (A.4),

$$\ker \mathrm{d}_{p;Y} = \Gamma\big(Y;\mathscr{Z}^p\big) = \check{H}^0\big(Y;\mathscr{Z}^p\big). \tag{A.5}$$

By [32, p193], each sheaf $\mathcal{S}_Y^p$ is fine. By [32, p202], this implies that

$$\check{H}^q(Y;\mathcal{S}_Y^p) = 0 \qquad \forall\ p\in\mathbb{Z}^{\geqslant 0},\ q\in\mathbb{Z}^+ . \tag{A.6}$$

Each $\mathbb{Z}_2$-valued singular p-cochain ϖ on Y determines a section $(\rho_{y,Y}(\varpi))_{y\in Y}$ of $\mathcal{S}_Y^p$ over Y. By [32, 5.32], the induced homomorphism

$$H^p(Y;\mathbb{Z}_2) \longrightarrow H^p\big(\Gamma(Y;\mathcal{S}_Y^*),\mathrm{d}_{*;Y}\big), \qquad [\varpi] \longrightarrow \big[(\rho_{y,Y}(\varpi))_{y\in Y}\big], \tag{A.7}$$

is an isomorphism. Combining the $p=0$ cases of this isomorphism and of the identification (A.5), we obtain an isomorphism (A.1) for $p=0$.

Suppose Y is locally H^k-simple and $p\in\mathbb{Z}^+$ with $p\leqslant k$. The sequence

$$\{0\} \longrightarrow \mathscr{Z}_Y^{p-q-1} \longrightarrow \mathcal{S}_Y^{p-q-1} \longrightarrow \mathscr{Z}_Y^{p-q} \longrightarrow \{0\} \tag{A.8}$$

of sheaves is then exact for every $q \in \mathbb{Z}^{\geqslant 0}$ with $q < p$. From the exactness of the associated long sequence in Čech cohomology, (A.5), and (A.6), we obtain isomorphisms

$$\begin{gathered}\check{\delta}_Y\colon H^p\big(\Gamma(Y;\mathcal{S}_Y^*),d_{*;Y}\big) \equiv \frac{\ker \mathrm{d}_{p;Y}}{\operatorname{Im}\mathrm{d}_{p-1;Y}} = \frac{\check{H}^0(Y;\mathscr{Z}^p)}{\mathrm{d}_{p-1}(\check{H}^0(Y;\mathcal{S}^{p-1}))} \xrightarrow{\approx} \check{H}^1\big(Y;\mathscr{Z}_Y^{p-1}\big),\\ \check{\delta}_Y\colon \check{H}^q(Y;\mathscr{Z}_Y^{p-q}) \xrightarrow{\approx} \check{H}^{q+1}\big(Y;\mathscr{Z}_Y^{p-q-1}\big)\ \ \forall\ q\in\mathbb{Z}^+,\ q<p.\end{gathered}$$

Putting these isomorphisms together, we obtain an isomorphism

$$\check{\delta}_Y^p\colon H^p\big(\Gamma(Y;\mathcal{S}_Y^*),d_{*;Y}\big) \longrightarrow \check{H}^p\big(Y;\mathscr{Z}_Y^0\big) = \check{H}^p\big(Y;\mathbb{Z}_2\big). \tag{A.9}$$

Combining (A.7) with this isomorphism, we obtain an isomorphism as in (A.1) with $p>0$.

Suppose Y is another paracompact locally H^k-simple space. A continuous map $f\colon Y \longrightarrow Y'$ induces commutative diagrams

$$\begin{array}{ccc} H^p(Y';\mathbb{Z}_2) & \xrightarrow{\approx} & H^p\big(\Gamma(Y';\mathcal{S}^*_{Y'}),d_{*;Y'}\big) \\ \Big\downarrow f^* & & \Big\downarrow f^* \\ H^p(Y;\mathbb{Z}_2) & \xrightarrow{\approx} & H^p\big(\Gamma(Y;\mathcal{S}^*_{Y}),d_{*;Y}\big) \end{array} \qquad \begin{array}{ccccccccc} \{0\} & \longrightarrow & \mathcal{Z}^{p-q-1}_{Y'} & \longrightarrow & \mathcal{S}^{p-q-1}_{Y'} & \longrightarrow & \mathcal{Z}^{p-q}_{Y'} & \longrightarrow & \{0\} \\ & & \Big\downarrow f^* & & \Big\downarrow f^* & & \Big\downarrow f^* & & \\ \{0\} & \longrightarrow & \mathcal{Z}^{p-q-1}_{Y} & \longrightarrow & \mathcal{S}^{p-q-1}_{Y} & \longrightarrow & \mathcal{Z}^{p-q}_{Y} & \longrightarrow & \{0\} \end{array}$$

for all $p,q\in\mathbb{Z}^{\geqslant 0}$ with $q<p\leqslant k$. Combining the $p=0$ cases of the first diagram above and of the identifications (A.5) for Y and for Y', we obtain a commutative diagram (A.2) for $p=0$. The second commutative diagram above induces a commutative diagram

$$\begin{array}{ccc} H^p\big(\Gamma(Y';\mathcal{S}^*_{Y'}),\mathrm{d}_{*;Y'}\big) & \xrightarrow{\approx} & \check{H}^p\big(Y';\mathbb{Z}_2\big) \\ \Big\downarrow f^* & & \Big\downarrow f^* \\ H^p\big(\Gamma(Y;\mathcal{S}^*_{Y}),\mathrm{d}_{*;Y}\big) & \xrightarrow{\approx} & \check{H}^p\big(Y;\mathbb{Z}_2\big) \end{array}$$

with the horizontal isomorphisms as in (A.9). Combining this with the first commutative diagram in this paragraph, we obtain (A.2) with $p>0$. □

Let Y be a paracompact locally H^1-simple space. We now describe the $p=1$ case of the isomorphism (A.1) explicitly. Suppose ϖ is a $\mathbb{Z}_2$-valued singular 1-cocycle on Y. Since Y is locally H^1-simple, there exist an open cover $\{U_\alpha\}_{\alpha\in\mathscr{A}}$ of Y and a $\mathbb{Z}_2$-valued singular 0-cochain μ_α on U_α for each $\alpha\in\mathscr{A}$ so that

$$\mathrm{d}_{0;U_\alpha}\mu_\alpha = \varpi\big|_{U_\alpha} \qquad \forall\,\alpha\in\mathscr{A}.$$

We define a Čech 1-cocycle η on Y by

$$\eta_{\alpha\beta} = \mu_\beta\big|_{U_{\alpha\beta}} - \mu_\alpha\big|_{U_{\alpha\beta}} \in \mathcal{S}^1_Y\big(U_{\alpha\beta}\big) \qquad \forall\,\alpha,\beta\in\mathscr{A}. \tag{A.10}$$

Since $\mathrm{d}_{0;U_{\alpha\beta}}\eta_{\alpha\beta}=0$ and Y is locally path-connected, $\eta_{\alpha\beta}$ is a locally constant function on $U_{\alpha\beta}$. Thus, η takes values in the sheaf of germs of $\mathbb{Z}_2$-valued continuous functions on Y and so defines an element $[\eta]$ of $\check{H}^1(Y;\mathbb{Z}_2)$. This is the image of $[\varpi]$ under the $p=1$ case of the isomorphism Φ_Y in (A.1).

Suppose Y is a CW complex and ϖ is a $\mathbb{Z}_2$-valued singular 1-cocycle on Y as above. For each vertex $\alpha\in Y_0$ of Y, let $U_\alpha\subset Y$ denote the (open) star of α, i.e. the union of all open cells $\mathring{e}$ of Y so that α is contained in the closed cell e. In particular, U_α is an open neighborhood of α and the collection $\{U_\alpha\}_{\alpha\in Y_0}$ covers the 1-skeleton $Y_1\subset Y$. We take

$$\{U_\alpha\}_{\alpha\in\mathscr{A}} \equiv \{U_\alpha\}_{\alpha\in Y_0} \sqcup \big\{Y-Y_1\big\};$$

this is an open cover of Y. By adding extra vertices to Y_1, we can ensure that no closed 1-cell is a cycle. This implies that every closed 1-cell e of Y runs between distinct vertices α and β with

$$e\subset U_\alpha\cup U_\beta, \qquad e\cap U_{\alpha\beta}=\mathring{e}, \qquad e\cap U_\gamma=\varnothing \;\;\forall\,\gamma\in\mathscr{A}-\{\alpha,\beta\}. \tag{A.11}$$

For every $\alpha\in\mathscr{A}$, there then exists a $\mathbb{Z}_2$-valued singular 0-cochain μ_α on U_α so that

$$\mathrm{d}_{0;U_\alpha}\mu_\alpha = \varpi\big|_{U_\alpha}, \quad \mu_\alpha(\alpha)=0 \qquad \forall\ \alpha\in\mathscr{A}.$$

Every closed 1-cell e of Y is cobordant to the difference of a singular 1-simplex $e_{x\beta}$ running from a point $x\in\mathring{e}$ to β and a singular 1-simplex $e_{x\alpha}$ running from x to α. By (A.11), $e_{x\alpha}\subset U_\alpha$ and $e_{x\beta}\subset U_\beta$. Since ϖ is a cocycle, it follows that

$$\begin{aligned}\varpi(e)=\varpi\big(e_{x\beta}-e_{x\alpha}\big)&=\varpi\big(e_{x\beta}\big)-\varpi\big(e_{x\alpha}\big)=\big\{\mathrm{d}_{0;U_\beta}\mu_\beta\big\}\big(e_{x\beta}\big)-\big\{\mathrm{d}_{0;U_\alpha}\mu_\alpha\big\}\big(e_{x\alpha}\big)\\&=\big(\mu_\beta(\beta)-\mu_\beta(x)\big)-\big(\mu_\alpha(\alpha)-\mu_\alpha(x)\big)=\mu_\alpha(x)-\mu_\beta(x).\end{aligned}$$

Along with (A.10), this implies that the Čech cohomology class $[\eta]\equiv\Phi_Y([\varpi])$ corresponding to $[\varpi]$ under the isomorphism (A.1) is represented by a collection $\{\eta_{\alpha\beta}\}_{\alpha,\beta\in\mathscr{A}}$ associated with an open cover $\{U_\alpha\}_{\alpha\in\mathscr{A}}$ of Y such that

$$\mathring{e}\subset U_{\alpha\beta}, \qquad \eta_{\alpha\beta}\big|_{\mathring{e}}=\varpi(e)\in\mathbb{Z}_2$$

for all $\alpha,\beta\in Y_0$ and every closed 1-cell e with vertices α and β.

A.2 Sheaves of groups

Čech cohomology groups $\check{H}^p$ are normally defined for sheaves or presheaves of (abelian) modules over a ring. The sets $\check{H}^0$ and $\check{H}^1$ can be defined for sheaves or presheaves of non-abelian groups as well. The first set is still a group, while the second is a pointed set, i.e. it has a distinguished element. A short exact sequence of such sheaves gives rise to an exact sequence of the corresponding Čech pointed sets, provided the kernel sheaf $\mathscr{R}$ lies in the center $Z(\mathscr{S})$ of the ambient sheaf $\mathscr{S}$; see Proposition A.3. The main examples of interest are the sheaves $\mathscr{S}$ of germs of continuous functions over a topological space Y with values in a Lie group G, as in Section A.3.

We denote the center of a group G by $Z(G)$. We call a collection

$$\big(\big(\delta_p\colon C^p\longrightarrow C^{p+1}\big)_{p=0,1,2},\ast\colon C^0\times C^1\longrightarrow C^1\big)$$

consisting of maps δ_p between groups C^p with the identity element $\mathbb{1}_p$ and a left action $*$ a short cochain complex if

$$\delta_p\mathbb{1}_p=\mathbb{1}_{p+1}, \quad \delta_{p+1}\circ\delta_p=\mathbb{1}_{p+2}, \quad \delta_1(f*g)=\delta_1 g \ \ \forall\, f\in C^0,\ g\in\delta_1^{-1}\big(Z(C^2)\big), \tag{A.12}$$

$$\delta_0(f\cdot f')=f*(\delta_0 f'), \quad f*g=(\delta_0 f)g \quad \forall\, f,f'\in C^0,\ g\in Z(C^1), \tag{A.13}$$

$$\delta_p\big(g\cdot g'\big)=\big(\delta_p g\big)\big(\delta_p g'\big) \quad \forall\, g\in C^p,\ g'\in Z(C^p),\ p=1,2. \tag{A.14}$$

By the second condition in (A.13),

$$C^0*\{g\}=\big(\mathrm{Im}\,\delta_0\big)\cdot\{g\} \quad \forall\ g\in Z(C^1). \tag{A.15}$$

By both conditions in (A.13),

$$H^0(C^*)\equiv H^0\big((C^p,\delta_p)_{p=0,1,2},*\big)\equiv\ker\delta_0\equiv\delta_0^{-1}(\mathbb{1}_1) \tag{A.16}$$

is a subgroup of C^0. By the last property in (A.12), $*$ restricts to an action on $\ker\delta_1\equiv\delta_1^{-1}(\mathbb{1}_2)$. We can thus define

$$H^1(C^*)\equiv H^1\big((C^p,\delta_p)_{p=0,1,2},*\big)=\ker\delta_1\big/C^0\,; \tag{A.17}$$

this is a pointed set with the distinguished element given by the image of $\mathrm{Im}\,\delta_0\ni\mathbb{1}_1$ in $H^1(C^*)$.

By (A.13) and the $p=1$ case of (A.14), δ_0 and δ_1 are group homomorphisms if the group C^1 is abelian and $*$ is the usual action of the 1-coboundaries on the 1-cochains via the group operation. In this case, (A.17) agrees with the usual definition and the last condition in (A.12) is automatically satisfied. If in addition the group C^2 is also abelian, as happens for the kernel complex B^* in Lemma A.2 below, then the map δ_2 is a group homomorphism as well and

$$H^2(C^*) \equiv H^2\big((C^p,\delta_p)_{p=0,1,2},*\big) = \ker\delta_2/\mathrm{Im}\,\delta_1$$

is a well-defined abelian group.

A morphism of short cochain complexes

$$\iota\equiv(\iota_p)_{p=0,1,2,3}\colon \big((B^p,\delta_p)_{p=0,1,2},*\big) \longrightarrow \big((C^p,\delta_p)_{p=0,1,2},*\big)$$

is a collection of group homomorphisms $\iota_p\colon B^p\longrightarrow C^p$ that commute with the maps δ_p and the actions $*$. Such a homomorphism induces morphisms

$$\iota_*\colon H^p(B^*)\longrightarrow H^p(C^*), \quad p=0,1,$$

of pointed sets, i.e. ι_* takes the distinguished element of the domain to the distinguished element of the target; the map ι_0 is a group homomorphism. The kernel of such a morphism is the preimage of the distinguished element of the target. The next lemma is an analogue of the Snake Lemma [32, Proposition 5.17] for short cochain complexes of groups.

Lemma A.2. *For every short exact sequence*

$$\{\mathbb{1}\} \longrightarrow \big((B^p,\delta_p)_{p=0,1,2},*\big) \xrightarrow{\ \iota\ } \big((C^p,\delta_p)_{p=0,1,2},*\big) \xrightarrow{\ \mathfrak{j}\ } \big((D^p,\delta_p)_{p=0,1,2},*\big) \longrightarrow \{\mathbb{1}\}$$

of short cochain complexes of groups such that $\iota_p(B^p)\subset Z(C^p)$ for $p=1,2$, there exist morphisms

$$\partial_p\colon H^p(D^*)\longrightarrow H^{p+1}(B^*), \quad p=0,1, \tag{A.18}$$

of pointed sets such that the sequence

$$\begin{aligned}\{\mathbb{1}\} \longrightarrow H^0(B^*) \xrightarrow{\iota_*} H^0(C^*) \xrightarrow{\mathfrak{j}_*} H^0(D^*) \xrightarrow{\partial_0} \\ \xrightarrow{\partial_0} H^1(B^*) \xrightarrow{\iota_*} H^1(C^*) \xrightarrow{\mathfrak{j}_*} H^1(D^*) \xrightarrow{\partial_1} H^2(B^*)\end{aligned} \tag{A.19}$$

of morphisms of pointed sets is exact. The maps ∂_p are natural with respect to morphisms of short exact sequences of short cochain complexes of groups.

Proof. We proceed as in the abelian case. Given $d_p\in\ker\delta_p\subset D^p$, let $c_p\in C^p$ be such that $\mathfrak{j}_p(c_p)=d_p$. Since

$$\mathfrak{j}_{p+1}\big(\delta_p(c_p)\big) = \delta_p\big(\mathfrak{j}_p(c_p)\big) = \delta_p(d_p) = \mathbb{1}_{p+1}\in D^{p+1},$$

there exists a unique $b_{p+1}\in B^{p+1}$ such that $\iota_{p+1}(b_{p+1})=\delta_p(c_p)$. By the second condition in (A.12), $b_{p+1}\in\ker\delta_{p+1}$. We set

$$\partial_p\big([d_p]\big) = \big[b_{p+1}\big]\in H^{p+1}(B^*).$$

By the first condition in (A.13), $[b_1]$ is independent of the choice of $c_0 \in C^0$ such that $\mathrm{j}_0(c_0)=d_0$. By the $p=1$ case of (A.14) and the assumption that $\iota_p(B^p)\subset Z(C^p)$ for $p=1,2$, $[b_2]$ is independent of the choice of $c_1 \in C^1$ such that $\mathrm{j}_1(c_1)=d_1$. By the last condition in (A.12) and the assumption that $\iota_2(B^2)\subset Z(C^2)$, $[b_2]$ does not depend on the choice of representative d_1 for $[d_1]$. Thus, the maps (A.18) are well-defined. By the first condition in (A.12), $\partial_p([\mathbb{1}_p])=[\mathbb{1}_{p+1}]$, i.e. ∂_p is a morphism of pointed sets. By the construction, the maps ∂_p are natural with respect to morphisms of exact sequences of short cochain complexes.

It is immediate that (A.19) is exact at $H^0(B^*)$ and $H^0(C^*)$ and that

$$\mathrm{j}_*\circ\iota_* = [\mathbb{1}_1]\colon H^1(B^*)\longrightarrow H^1(D^*), \quad \partial_p\circ\mathrm{j}_* = [\mathbb{1}_{p+1}]\colon H^p(C^*)\longrightarrow H^{p+1}(B^*)\,.$$

The exactness of (A.19) at $H^1(B^*)$ is immediate from (A.15) with $g=\mathbb{1}_1\in Z(C^1)$. The exactness at $H^1(C^*)$ follows from (A.15) with $g=\mathbb{1}_1\in Z(D^1)$, the second condition in (A.13), and the assumption that $\iota_1(B^1)\subset Z(C^1)$. The exactness at $H^0(D^*)$ follows from (A.15) with $g=\mathbb{1}_1\in Z(B^1)$, both conditions in (A.13), and the assumption that $\iota_1(B^1)\subset Z(C^1)$. The exactness at $H^1(D^*)$ follows from the assumption that $\iota_p(B^p)\subset Z(C^p)$ for $p=1,2$ and the $p=1$ case of (A.14). □

We next review the definitions and key properties of the group $\check{H}^0$ and pointed set $\check{H}^1$ for a sheaf $\mathcal{S}$ of groups over a topological space Y. We denote by $Z(\mathcal{S})\subset\mathcal{S}$ the subsheaf consisting of the centers $Z(\mathcal{S}_y)$ of the groups $\mathcal{S}_y$ with $y\in Y$ and by $\mathbb{1}_y\in\mathcal{S}_y$ the identity element of $\mathcal{S}_y$.

Let $\underline{U}\equiv\{U_\alpha\}_{\alpha\in\mathcal{A}}$ be an open cover of Y. As in the abelian case, the set

$$\check{C}^p(\underline{U};\mathcal{S}) \equiv \prod_{\alpha_0,\alpha_1,\ldots,\alpha_p\in\mathcal{A}} \Gamma\big(U_{\alpha_0\alpha_1\ldots\alpha_p};\mathcal{S}\big)$$

of Čech p-cochains is a group under pointwise multiplication of sections:

$$\begin{gathered}\cdot : \check{C}^p(\underline{U};\mathcal{S})\times\check{C}^p(\underline{U};\mathcal{S})\longrightarrow\check{C}^p(\underline{U};\mathcal{S}),\\ \{h\cdot h'\}_{\alpha_0\alpha_1\ldots\alpha_p}(y) = h_{\alpha_0\alpha_1\ldots\alpha_p}(y)\cdot h'_{\alpha_0\alpha_1\ldots\alpha_k}(y) \quad \forall\,\alpha_0,\alpha_1,\ldots,\alpha_p\in\mathcal{A},\ y\in U_{\alpha_0\alpha_1\ldots\alpha_p}\,.\end{gathered}$$

The identity element $\mathbb{1}_p\in\check{C}^p(\underline{U};\mathcal{S})$ is given by

$$(\mathbb{1}_p)_{\alpha_0\alpha_1\ldots\alpha_p}(y) = \mathbb{1}_y \qquad \forall\,\alpha_0,\alpha_1,\ldots,\alpha_p\in\mathcal{A},\ y\in U_{\alpha_0\alpha_1\ldots\alpha_p}\,.$$

Define the boundary maps by

$$\begin{gathered}\delta_0\colon\check{C}^0(\underline{U};\mathcal{S})\longrightarrow\check{C}^1(\underline{U};\mathcal{S}), \quad (\delta_0 f)_{\alpha_0\alpha_1} = f_{\alpha_0}\big|_{U_{\alpha_0\alpha_1}}\cdot f^{-1}_{\alpha_1}\big|_{U_{\alpha_0\alpha_1}}\,,\\ \delta_1\colon\check{C}^1(\underline{U};\mathcal{S})\longrightarrow\check{C}^2(\underline{U};\mathcal{S}), \quad (\delta_1 g)_{\alpha_0\alpha_1\alpha_2} = g_{\alpha_1\alpha_2}\big|_{U_{\alpha_0\alpha_1\alpha_2}}\cdot g^{-1}_{\alpha_0\alpha_2}\big|_{U_{\alpha_0\alpha_1\alpha_2}}\cdot g_{\alpha_0\alpha_1}\big|_{U_{\alpha_0\alpha_1\alpha_2}}\,.\end{gathered}$$

We also define a left action of $\check{C}^0(\underline{U};\mathcal{S})$ on $\check{C}^1(\underline{U};\mathcal{S})$ by

$$\begin{gathered}*\colon\check{C}^0(\underline{U};\mathcal{S})\times\check{C}^1(\underline{U};\mathcal{S})\longrightarrow\check{C}^1(\underline{U};\mathcal{S}),\\ \{f*g\}_{\alpha_0\alpha_1} = f_{\alpha_0}\big|_{U_{\alpha_0\alpha_1}}\cdot g_{\alpha_0\alpha_1}\cdot f^{-1}_{\alpha_1}\big|_{U_{\alpha_0\alpha_1}}\in\Gamma(U_{\alpha_0\alpha_1};\mathcal{S}).\end{gathered}$$

We now construct a short cochain complex. Let

$$C^p\big(\underline{U};\mathcal{S}\big)=\begin{cases}\check{C}^p(\underline{U};\mathcal{S}), & \text{if } p=0,1,2;\\ \mathrm{Abel}(\check{C}^p(\underline{U};\mathcal{S})) & \text{if } p=3.\end{cases}$$

For $p=0,1$, we take

$$\delta_p\colon C^p\big(\underline{U};\mathcal{S}\big)\longrightarrow C^{p+1}\big(\underline{U};\mathcal{S}\big)$$

to be the maps defined above. We take δ_2 to be the composition of the map

$$\delta_2\colon \check{C}^2(\underline{U};\mathcal{S})\longrightarrow \check{C}^3(\underline{U};\mathcal{S}),$$
$$\big(\delta_2 h\big)_{\alpha_0\alpha_1\alpha_2\alpha_3}=h_{\alpha_1\alpha_2\alpha_3}\big|_{U_{\alpha_0\alpha_1\alpha_2\alpha_3}}h^{-1}_{\alpha_0\alpha_2\alpha_3}\big|_{U_{\alpha_0\alpha_1\alpha_2\alpha_3}}h_{\alpha_0\alpha_1\alpha_3}\big|_{U_{\alpha_0\alpha_1\alpha_2\alpha_3}}h^{-1}_{\alpha_0\alpha_1\alpha_2}\big|_{U_{\alpha_0\alpha_1\alpha_2\alpha_3}},$$

with the projection $\check{C}^3(\underline{U};\mathcal{S})\longrightarrow C^3\big(\underline{U};\mathcal{S}\big)$. The tuple

$$\Big(\big(\delta_p\colon C^p(\underline{U};\mathcal{S})\longrightarrow C^{p+1}(\underline{U};\mathcal{S})\big)_{p=0,1,2},\ast\colon C^0(\underline{U};\mathcal{S})\times C^1(\underline{U};\mathcal{S})\longrightarrow C^1(\underline{U};\mathcal{S})\Big)$$

is then a short cochain complex of groups. We denote the associated group (A.16) and the pointed set (A.17) by $\check{H}^0(\underline{U};\mathcal{S})$ and $\check{H}^1(\underline{U};\mathcal{S})$, respectively.

Let $\underline{U}'\equiv\{U'_\alpha\}_{\alpha\in\mathscr{A}'}$ be an open cover of Y refining $\underline{U}$, i.e. there exists a map $\mu\colon\mathscr{A}'\longrightarrow\mathscr{A}$ such that $U'_\alpha\subset U_{\mu(\alpha)}$ for every $\alpha\in\mathscr{A}'$. Such a refining map induces group homomorphisms

$$\mu_p^*\colon \check{C}^p(\underline{U};\mathcal{S})\longrightarrow \check{C}^p(\underline{U}';\mathcal{S}), \tag{A.20}$$
$$\big(\mu_p^*h\big)_{\alpha_0\ldots\alpha_p}=h_{\mu(\alpha_0)\ldots\mu(\alpha_p)}\big|_{U'_{\alpha_0\ldots\alpha_p}}\quad\forall\ h\in\check{C}^p(\underline{U};\mathcal{S}),\ \alpha_0,\ldots,\alpha_p\in\mathscr{A}'.$$

These homomorphisms commute with δ_0, δ_1, and the action of $\check{C}^0(\cdot;\mathcal{S})$ on $\check{C}^1(\cdot;\mathcal{S})$. Thus, μ induces maps

$$R^0_{\underline{U}',\underline{U}}\colon \check{H}^0(\underline{U};\mathcal{S})\longrightarrow\check{H}^0(\underline{U}';\mathcal{S})\quad\text{and}\quad R^1_{\underline{U}',\underline{U}}\colon \check{H}^1(\underline{U};\mathcal{S})\longrightarrow\check{H}^1(\underline{U}';\mathcal{S}) \tag{A.21}$$

of pointed sets; the first map above is a group homomorphism.

If $\mu'\colon\mathscr{A}'\longrightarrow\mathscr{A}$ is another refining map, then $U'_\alpha\subset U_{\mu(\alpha)\mu'(\alpha)}$ for every $\alpha\in\mathscr{A}'$ and thus

$$\mu_0^*\big|_{\ker\delta_0}=\mu_0'^*\big|_{\ker\delta_0}\colon\ker\delta_0\longrightarrow\ker\delta_0\subset\check{C}^0(\underline{U}';\mathcal{S}).$$

For $g\in\check{C}^1(\underline{U};\mathcal{S})$, define

$$h_1g\in\check{C}^0(\underline{U}';\mathcal{S})\qquad\text{by}\qquad (h_1g)_\alpha=g_{\mu'(\alpha)\mu(\alpha)}|_{U'_\alpha}.$$

If $g\in\ker\delta_1\subset\check{C}^1(\underline{U};\mathcal{S})$, then

$$g_{\mu(\alpha_0)\mu(\alpha_1)}\big|_{U_{\mu'(\alpha_1)\mu(\alpha_0)\mu(\alpha_1)}}\cdot g^{-1}_{\mu'(\alpha_1)\mu(\alpha_1)}\big|_{U_{\mu'(\alpha_1)\mu(\alpha_0)\mu(\alpha_1)}}=g^{-1}_{\mu'(\alpha_1)\mu(\alpha_0)}\big|_{U_{\mu'(\alpha_1)\mu(\alpha_0)\mu(\alpha_1)}},$$
$$g_{\mu'(\alpha_0)\mu(\alpha_0)}\big|_{U_{\mu'(\alpha_1)\mu'(\alpha_0)\mu(\alpha_0)}}\cdot g^{-1}_{\mu'(\alpha_1)\mu(\alpha_0)}\big|_{U_{\mu'(\alpha_1)\mu'(\alpha_0)\mu(\alpha_0)}}=g^{-1}_{\mu'(\alpha_1)\mu'(\alpha_0)}\big|_{U_{\mu'(\alpha_1)\mu'(\alpha_0)\mu(\alpha_0)}}$$

for all $\alpha_0,\alpha_1\in\mathscr{A}$. From this, we find that

$$\mu_1'^*g=(h_1g)\ast(\mu_1^*g)\quad\forall\ g\in\ker\delta_1\subset\check{C}^1(\underline{U};\mathcal{S}).$$

By the previous paragraph, the pointed maps (A.21) are independent of the choice of refining map $\mu\colon \mathscr{A}'\longrightarrow\mathscr{A}$. We can therefore define the group $\check{H}^0(Y;\mathcal{S})$ and the pointed set $\check{H}^1(Y;\mathcal{S})$ as the direct limits of the groups $\check{H}^0(\underline{U};\mathcal{S})$ and of the pointed sets $\check{H}^1(\underline{U};\mathcal{S})$, respectively, over open covers of Y. The map

$$\Gamma(Y;\mathcal{S}) \longrightarrow \check{H}^0(Y;\mathcal{S}), \qquad f \longrightarrow \big(f|_{U_\alpha}\big)_{\alpha\in\mathscr{A}}, \tag{A.22}$$

is a group isomorphism.

If $\mathcal{S}$ is a sheaf of abelian groups, as happens for the kernel sheaf $\mathcal{R}$ in Proposition A.3 below, the definitions of $\check{H}^0(Y;\mathcal{S})$ and $\check{H}^1(Y;\mathcal{S})$ above reduce to the ones in [32, Section 5.33]. Furthermore,

$$\check{H}^2(\underline{U};\mathcal{S}) \equiv \frac{\ker(\delta_2\colon \check{C}^2(\underline{U};\mathcal{S})\longrightarrow\check{C}^3(\underline{U};\mathcal{S}))}{\mathrm{Im}(\delta_1\colon \check{C}^1(\underline{U};\mathcal{S})\longrightarrow\check{C}^2(\underline{U};\mathcal{S}))}$$

is a well-defined abelian group for every open cover $\underline{U}$ of Y. The group homomorphisms

$$R^2_{\underline{U}',\underline{U}}\colon \check{H}^2(\underline{U};\mathcal{S}) \longrightarrow \check{H}^2(\underline{U}';\mathcal{S})$$

induced by refining maps still depend only on the covers $\underline{U}$ and $\underline{U}'$. The abelian group $\check{H}^2(Y;\mathcal{S})$ is again the direct limit of the groups $\check{H}^2(\underline{U};\mathcal{S})$ over all open covers $\underline{U}$ of Y.

A homomorphism $\iota\colon\mathcal{R}\longrightarrow\mathcal{S}$ of sheaves of groups over Y induces maps

$$\iota_*\colon \Gamma(Y;\mathcal{R}) \longrightarrow \Gamma(Y;\mathcal{S}), \quad \iota_*\colon \check{H}^0(Y;\mathcal{R}) \longrightarrow \check{H}^0(Y;\mathcal{S}), \quad \iota_*\colon \check{H}^1(Y;\mathcal{R}) \longrightarrow \check{H}^1(Y;\mathcal{S})$$

between pointed spaces. The first two maps are group homomorphisms which commute with the identifications (A.22).

Proposition A.3. *Let Y be a paracompact space. For every short exact sequence*

$$\{\mathbb{1}\} \longrightarrow \mathcal{R} \xrightarrow{\ \iota\ } \mathcal{S} \xrightarrow{\ \mathfrak{j}\ } \mathcal{T} \longrightarrow \{\mathbb{1}\} \tag{A.23}$$

of sheaves of groups over Y such that $\iota(\mathcal{R})\subset Z(\mathcal{S})$, there exist morphisms

$$\check{\delta}_p\colon \check{H}^p(Y;\mathcal{T}) \longrightarrow \check{H}^{p+1}(Y;\mathcal{R}), \quad p=0,1, \tag{A.24}$$

of pointed sets such that the sequence

$$\begin{aligned}\{\mathbb{1}\} \longrightarrow \check{H}^0(Y;\mathcal{R}) \xrightarrow{\iota_*} \check{H}^0(Y;\mathcal{S}) \xrightarrow{\mathfrak{j}_*} \check{H}^0(Y;\mathcal{T}) \xrightarrow{\check{\delta}_0} \\ \xrightarrow{\check{\delta}_0} \check{H}^1(Y;\mathcal{R}) \xrightarrow{\iota_*} \check{H}^1(Y;\mathcal{S}) \xrightarrow{\mathfrak{j}_*} \check{H}^1(Y;\mathcal{T}) \xrightarrow{\check{\delta}_1} \check{H}^2(Y;\mathcal{R})\end{aligned} \tag{A.25}$$

of morphisms of pointed sets is exact. The maps $\check{\delta}_p$ are natural with respect to morphisms of short exact sequences of sheaves of groups over Y.

Proof. Let $\underline{U}\equiv\{U_\alpha\}_{\alpha\in\mathscr{A}}$ be an open cover of Y,

$$B^p(\underline{U}) = C^p\big(\underline{U};\mathcal{R}\big), \quad C^p(\underline{U}) = C^p\big(\underline{U};\mathcal{S}\big), \quad D^p(\underline{U}) = C^p\big(\underline{U};\mathcal{T}\big).$$

Since $\iota(\mathscr{R})\subset Z(\mathscr{S})$, $\iota_*(B^p(\underline{U}))\subset Z(C^p(\underline{U}))$ for all p. By the exactness of (A.23), the sequence

$$\{\mathbb{1}\}\longrightarrow B^p(\underline{U})\xrightarrow{\iota_*}C^p(\underline{U})\xrightarrow{\mathfrak{j}_*}D^p(\underline{U})$$

of groups is exact. For $p=0,1,2,3$, we denote by $\overline{D}^p(\underline{U})\subset D^p(\underline{U})$ the image of $\mathfrak{j}_*$. For $p=0,1$, let $\overline{H}^p(\underline{U};\mathscr{T})$ be the Čech pointed sets determined by the short cochain complex $\overline{D}^*(\underline{U})$.

The sequence

$$\{\mathbb{1}\}\longrightarrow B^p(\underline{U})\xrightarrow{\iota_*}C^p(\underline{U})\xrightarrow{\mathfrak{j}_*}\overline{D}^p(\underline{U})\longrightarrow\{\mathbb{1}\} \tag{A.26}$$

of short cochain complexes is exact. By Lemma A.2, there thus exist morphisms

$$\check{\delta}_p\colon \overline{H}^p(\underline{U};\mathscr{T})\longrightarrow \check{H}^{p+1}(\underline{U};\mathscr{R}),\quad p=0,1, \tag{A.27}$$

of pointed sets such that the sequence

$$\begin{split}\{\mathbb{1}\}\longrightarrow \check{H}^0(\underline{U};\mathscr{R})\xrightarrow{\iota_*}\check{H}^0(\underline{U};\mathscr{S})\xrightarrow{\mathfrak{j}_*}\overline{H}^0(\underline{U};\mathscr{T})\xrightarrow{\check{\delta}_0}\\ \xrightarrow{\check{\delta}_0}\check{H}^1(\underline{U};\mathscr{R})\xrightarrow{\iota_*}\check{H}^1(\underline{U};\mathscr{S})\xrightarrow{\mathfrak{j}_*}\check{H}^1(\underline{U};\mathscr{T})\xrightarrow{\check{\delta}_1}\check{H}^2(\underline{U};\mathscr{R})\end{split} \tag{A.28}$$

of morphisms of pointed sets is exact.

Let $\underline{U}'\equiv\{U'_\alpha\}_{\alpha\in\mathscr{A}'}$ be an open cover of Y refining $\underline{U}$ and $\mu\colon\mathscr{A}'\longrightarrow\mathscr{A}$ be a refining map. By the naturality of the morphisms (A.27), the group homomorphisms (A.20) induce commutative diagrams

$$\begin{array}{ccc}\overline{H}^p(\underline{U};\mathscr{T}) & \xrightarrow{\quad\check{\delta}_p\quad} & \check{H}^{p+1}(\underline{U};\mathscr{R})\\ {\scriptstyle R^p_{\underline{U}',\underline{U}}}\big\downarrow & & \big\downarrow{\scriptstyle R^{p+1}_{\underline{U}',\underline{U}}}\\ \overline{H}^p(\underline{U}';\mathscr{T}) & \xrightarrow{\quad\check{\delta}_p\quad} & \check{H}^{p+1}(\underline{U}';\mathscr{R})\end{array}$$

of pointed sets. Taking the direct limit of the morphisms (A.27) over all open covers of Y, we thus obtain morphisms

$$\check{\delta}_p\colon \overline{H}^p(Y;\mathscr{T})\longrightarrow \check{H}^{p+1}(Y;\mathscr{R}),\quad p=0,1, \tag{A.29}$$

of pointed sets such that the sequence (A.25) with $\check{H}^*(Y;\mathscr{T})$ replaced by $\overline{H}^{p+1}(Y;\mathscr{T})$ is exact.

The inclusions $\mathfrak{i}_p\colon\overline{D}^p(\underline{U})\longrightarrow D^p(\underline{U})$ of short cochain complexes commute with the refining homomorphisms (A.20) and induce morphisms

$$\mathfrak{i}_*\colon\overline{H}^p(\underline{U};\mathscr{T})\longrightarrow\check{H}^p(\underline{U};\mathscr{T})\qquad\text{and}\qquad \mathfrak{i}_*\colon\overline{H}^p(Y;\mathscr{T})\longrightarrow\check{H}^p(Y;\mathscr{T}) \tag{A.30}$$

of pointed sets. By the paracompactness of Y and the reasoning in [32, p204], for every open cover $\underline{U}\equiv\{U_\alpha\}_{\alpha\in\mathscr{A}}$ of Y and every element d_p of $D^p(\underline{U})$ there exist an open cover $\underline{U}'\equiv\{U'_\alpha\}_{\alpha\in\mathscr{A}'}$ refining $\underline{U}$, a refining map $\mu\colon\mathscr{A}'\longrightarrow\mathscr{A}$, and an element d'_p of $\overline{D}^p(\underline{U}')$ such that $\mathfrak{i}_p(d'_p)=\mu_p^*(d_p)$. This implies that the second map in (A.30) is a bijection. Composing (A.29) with this bijection, we obtain a morphism as in (A.24) so that the sequence (A.25) is exact.

A morphism of short exact sequences of sheaves of groups over Y as in (A.23) induces morphisms of the corresponding exact sequences of short cochain complexes as in (A.26) and of the inclusions $\mathfrak{i}_p$

above. Thus, it also induces morphisms of the corresponding maps as in (A.27) and as on the left-hand side of (A.30). These morphisms commute with the associated maps (A.21) and thus induce morphisms of the maps as in (A.24). This establishes the last claim. □

A.3 Sheaves determined by Lie groups

For a Lie group G and a topological space Y, we denote by $\mathcal{S}_Y(G)$ the sheaf of germs of continuous G-valued functions on Y and let

$$\check{H}^p(Y;G) = \check{H}^p\big(Y;\mathcal{S}_Y(G)\big) \qquad \forall\ p=0,1.$$

If G is abelian, we use the same notation for all $p\in\mathbb{Z}$. We begin this section by applying Proposition A.3 to short exact sequences of sheaves arising from short exact sequences

$$\{\mathbb{1}\} \longrightarrow K \stackrel{\iota}{\longrightarrow} G \stackrel{\mathfrak{j}}{\longrightarrow} Q \longrightarrow \{\mathbb{1}\} \tag{A.31}$$

of Lie groups. For certain kinds of exact sequences (A.31), the topological condition on Y of Definition A.4 appearing in the resulting statement of Corollary A.5 reduces to the locally H^1-simple notion of Definition 3.2. For such exact sequences of Lie groups and topological spaces, we combine Proposition A.1 and Corollary A.5 to obtain an exact sequence mixing Čech and singular cohomology; see Proposition A.6.

A homomorphism $\iota\colon K\longrightarrow G$ of Lie groups induces a homomorphism

$$\iota\colon \mathcal{S}_Y(K) \longrightarrow \mathcal{S}_Y(G)$$

of sheaves over every topological space and thus morphisms

$$\iota_*\colon \check{H}^p(Y;K) \longrightarrow \check{H}^p(Y;G)$$

of pointed sets for $p=0,1$; the $p=0$ case of ι_* is a group homomorphism.

A continuous map $f\colon Y\longrightarrow Y'$ induces group homomorphisms

$$f^*\colon \check{C}^p\big(\underline{U};\mathcal{S}_{Y'}(G)\big) \longrightarrow \check{C}^p\big(f^{-1}(\underline{U});\mathcal{S}_Y(G)\big), \qquad p\in\mathbb{Z},$$

for every open cover $\underline{U}$ of Y' that commute with the Čech coboundaries and group actions for the sheaves $\mathcal{S}_{Y'}(G)$ and $\mathcal{S}_Y(G)$ constructed in Section A.2 and with the refining homomorphisms as in (A.20). Thus, f induces morphisms

$$f^*\colon \check{H}^p(Y';G) \longrightarrow \check{H}^p(Y;G)$$

of pointed sets for $p=0,1$; the $p=0$ case of f^* is a group homomorphism. If G is abelian, then f induces such a morphism for every $p\in\mathbb{Z}$ and this morphism is a group homomorphism. If in addition ι is a homomorphism of Lie groups as above, then the diagram

$$\begin{array}{ccc} \check{H}^p(Y';K) & \xrightarrow{\ \iota_*\ } & \check{H}^p(Y';G) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \check{H}^p(Y;K) & \xrightarrow{\ \iota_*\ } & \check{H}^p(Y;G) \end{array}$$

commutes.

Definition A.4. Let (A.31) be a short exact sequence of Lie groups. A topological space Y is locally simple with respect to (A.31) if it is locally path-connected and for every neighborhood $U\subset Y$ of a point $y\in Y$ and a continuous map $f_U: U\longrightarrow Q$ there exist a neighborhood $U'\subset U$ of y and a continuous map $f'_U: U'\longrightarrow G$ such that $f_U|_{U'}=\mathfrak{j}\circ f'_U$.

For any topological space Y, a short exact sequence (A.31) of Lie groups induces an exact sequence

$$\{\mathbb{1}\}\longrightarrow \mathcal{S}_Y(K)\stackrel{\iota}{\longrightarrow}\mathcal{S}_Y(G)\stackrel{\mathfrak{j}}{\longrightarrow}\mathcal{S}_Y(Q)$$

of sheaves over Y. The last map above is surjective if and only if Y is locally simple with respect to (A.31). If the restriction of $\mathfrak{j}$ to the identity component G_0 of G is a double cover of Q_0 and $\pi_1(Q_0)$ is (possibly infinite) cyclic, then the condition of Definition A.4 is equivalent to Y being locally H^1-simple. This follows from the lifting property for covering projections [27, Lemma 79.1], Hurewicz isomorphism for π_1 [31, Proposition 7.5.2], and the Universal Coefficient Theorem for Cohomology [26, Theorem 53.3].

Corollary A.5. *Let Y be a paracompact space and (A.31) be a short exact sequence of Lie groups such that $\iota(K)\subset Z(G)$. If Y is locally simple with respect to (A.31), then there exist morphisms*

$$\check{\delta}_p\colon \check{H}^p(Y;Q)\longrightarrow \check{H}^{p+1}(Y;K),\quad p=0,1, \tag{A.32}$$

of pointed sets such that the sequence

$$\begin{aligned}\{\mathbb{1}\}\longrightarrow \check{H}^0(Y;K)\stackrel{\iota_*}{\longrightarrow}\check{H}^0(Y;G)\stackrel{\mathfrak{j}_*}{\longrightarrow}\check{H}^0(Y;Q)\stackrel{\check{\delta}_0}{\longrightarrow}\\ \stackrel{\check{\delta}_0}{\longrightarrow}\check{H}^1(Y;K)\stackrel{\iota_*}{\longrightarrow}\check{H}^1(Y;G)\stackrel{\mathfrak{j}_*}{\longrightarrow}\check{H}^1(Y;Q)\stackrel{\check{\delta}_1}{\longrightarrow}\check{H}^2(Y;K)\end{aligned} \tag{A.33}$$

of morphisms of pointed sets is exact. The maps $\check{\delta}_p$ are natural with respect to morphisms of short exact sequences of Lie groups and with respect to continuous maps between paracompact spaces that are locally simple with respect to (A.31).

Proof. Since Y is locally simple with respect to (A.31), the sequence

$$\{\mathbb{1}\}\longrightarrow \mathcal{S}_Y(K)\stackrel{\iota}{\longrightarrow}\mathcal{S}_Y(G)\stackrel{\mathfrak{j}}{\longrightarrow}\mathcal{S}_Y(Q)\longrightarrow\{\mathbb{1}\} \tag{A.34}$$

of sheaves over Y is exact. Since $\iota(K)\subset Z(G)$, $\iota(\mathcal{S}_Y(K))\subset Z(\mathcal{S}_Y(G))$. The existence of morphisms (A.32) so that the sequence (A.33) is exact thus follows from the first statement of Proposition A.3.

A morphism of short exact sequences of Lie groups as in (A.31) satisfying the conditions at the beginning of the statement of the proposition induces a morphism of the corresponding short exact sequences of sheaves as in (A.34). Thus, the naturality of (A.32) with respect to morphisms of short exact sequences of Lie groups follows from the second statement of Proposition A.3.

A continuous map $f: Y\longrightarrow Y'$ between paracompact spaces that are locally simple with respect to (A.31) induces a morphism of the corresponding exact sequences of short cochain complexes as in (A.26) and of the inclusions $\mathfrak{i}_p$ as in the proof of Proposition A.3. Thus, it also induces morphisms of the corresponding maps as in (A.27) and as on the left-hand side of (A.30). These morphisms commute with the associated maps (A.21) and thus induce morphisms of the maps as in (A.32). This establishes the naturality of (A.32) with respect to continuous maps. □

Proposition A.6. *Let Y be a paracompact locally H^1-simple space and*

$$\{\mathbb{1}\} \longrightarrow \mathbb{Z}_2 \xrightarrow{\iota} G \xrightarrow{\mathrm{j}} Q \longrightarrow \{\mathbb{1}\} \tag{A.35}$$

be an exact sequence of Lie groups such that $\iota(\mathbb{Z}_2)\subset Z(G)$ and $\pi_1(Q_0)$ is cyclic. Then there exist morphisms

$$\check{\delta}_0 : \check{H}^0(Y;Q) \longrightarrow H^1(Y;\mathbb{Z}_2) \quad \textit{and} \quad \check{\delta}_1 : \check{H}^1(Y;Q) \longrightarrow \check{H}^2(Y;\mathbb{Z}_2), \tag{A.36}$$

of pointed sets such that the sequence

$$\begin{aligned}\{\mathbb{1}\} \longrightarrow H^0(Y;\mathbb{Z}_2) \xrightarrow{\iota_*} \check{H}^0(Y;G) \xrightarrow{\mathrm{j}_*} \check{H}^0(Y;Q) \xrightarrow{\check{\delta}_0} \\ \xrightarrow{\check{\delta}_0} H^1(Y;\mathbb{Z}_2) \xrightarrow{\iota_*} \check{H}^1(Y;G) \xrightarrow{\mathrm{j}_*} \check{H}^1(Y;Q) \xrightarrow{\check{\delta}_1} \check{H}^2(Y;\mathbb{Z}_2)\end{aligned} \tag{A.37}$$

of morphisms of pointed sets is exact. If in addition Y is locally H^2-simple, then the same statement with $\check{H}^2(Y;\mathbb{Z}_2)$ replaced by $H^2(Y;\mathbb{Z}_2)$ also holds. The maps $\check{\delta}_0$ and $\check{\delta}_1$ are natural with respect to morphisms of exact sequences of Lie groups as in (A.6) and with respect to continuous maps between paracompact locally H^1-simple spaces.

Proof. Since Y is locally H^1-simple, it is locally simple with respect to the exact sequence (A.35) in the sense of Definition A.4. Thus, this proposition with all $H^p(Y;\mathbb{Z}_2)$ replaced by $\check{H}^p(Y;\mathbb{Z}_2)$ is a specialization of Corollary A.5. By Proposition A.1, we can then replace $\check{H}^0(Y;\mathbb{Z}_2)$ by $H^0(Y;\mathbb{Z}_2)$ and $\check{H}^1(Y;\mathbb{Z}_2)$ by $H^1(Y;\mathbb{Z}_2)$. If in addition Y is locally H^2-simple, then $\check{H}^2(Y;\mathbb{Z}_2)$ can also be replaced by $H^2(Y;\mathbb{Z}_2)$. □

A.4 Relation with principal bundles

Let G be a Lie group and Y be a topological space. We recall below the standard identification of the set $\mathrm{Prin}_Y(G)$ of equivalence (isomorphism) classes of principal G-bundles over Y with the pointed set $\check{H}^1(Y;G)$. This identification is key for applying Proposition A.6 to principal G-bundles, including to study Spin- and Pin$^{\pm}$-structures in the classical perspective of Definition 1.1.

Suppose $\pi_P : P \longrightarrow Y$ is a principal G-bundle. Let $\underline{U} \equiv (U_\alpha)_{\alpha\in\mathscr{A}}$ be an open cover of Y so that the principal G-bundle $P|_{U_\alpha}$ is trivializable for every $\alpha\in\mathscr{A}$. Thus, for every $\alpha\in\mathscr{A}$ there exists a homeomorphism

$$\begin{gathered}\Phi_\alpha : P\big|_{U_\alpha} \longrightarrow U_\alpha \times G \qquad \text{s.t.} \\ \pi_{\alpha;1}\circ\Phi_\alpha = \pi_P, \quad \pi_{\alpha;2}\big(\Phi_\alpha(pu)\big) = \big(\pi_{\alpha;2}\big(\Phi_\alpha(p)\big)\big)\cdot u \quad \forall\, p\in P|_{U_\alpha},\, u\in G,\end{gathered}$$

where $\pi_{\alpha;1}, \pi_{\alpha;2} : U_\alpha \times G \longrightarrow U_\alpha, G$ are the two projection maps. Thus, for all $\alpha,\beta\in\mathscr{A}$ there exists a continuous map

$$g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow G \qquad \text{s.t.} \quad \pi_{\alpha;2}\big(\Phi_\alpha(p)\big) = g_{\alpha\beta}\big(\pi_P(p)\big)\cdot\big(\pi_{\beta;2}\big(\Phi_\beta(p)\big)\big) \quad \forall\, p\in P\big|_{U_{\alpha\beta}}.$$

These continuous maps satisfy

$$g_{\beta\gamma}\big|_{U_{\alpha\beta\gamma}}\cdot g_{\alpha\gamma}^{-1}\big|_{U_{\alpha\beta\gamma}}\cdot g_{\alpha\beta}\big|_{U_{\alpha\beta\gamma}} = \mathbb{1} \qquad \forall\, \alpha,\beta,\gamma\in\mathscr{A}.$$

Therefore, $g_P \equiv (g_{\alpha\beta})_{\alpha,\beta\in\mathcal{A}}$ lies in $\ker\delta_1 \subset \check{C}^1(\underline{U};\mathcal{S}_Y(G))$ and thus defines an element

$$[g_P] \in \check{H}^1(Y;G).$$

We show below that $[g_P]$ depends only on the isomorphism class of P.

Suppose $\underline{U}' \equiv \{U'_\alpha\}_{\alpha\in\mathcal{A}'}$ is a refinement of $\underline{U}$. If $\mu\colon \mathcal{A}' \longrightarrow \mathcal{A}$ is a refining map, then

$$\Phi_\alpha \equiv \Phi_{\mu(\alpha)}\big|_{P|_{U'_\alpha}} : P\big|_{U'_\alpha} \longrightarrow U'_\alpha \times G$$

is a trivialization of the principal G-bundle $P|_{U'_\alpha}$ for every $\alpha \in \mathcal{A}'$. The corresponding transition data is

$$g'_P \equiv \big\{g_{\mu(\alpha)\mu(\beta)}|_{U'_{\alpha\beta}} : U'_{\alpha\beta} \longrightarrow G\big\}_{\alpha,\beta\in\mathcal{A}'} = \mu_1^* g_P.$$

Since

$$[g_P] = [\mu_1^* g_P] \in \check{H}^1(Y;G),$$

it is thus sufficient to consider trivializations of isomorphic vector bundles over a common cover (otherwise we can simply take the intersections of open sets in the two covers).

Suppose $\Psi\colon P \longrightarrow P'$ is an isomorphism of principal G-bundles over Y and the principal G-bundle $P'|_{U_\alpha}$ is trivializable for every $\alpha\in\mathcal{A}$. Thus, for every $\alpha\in\mathcal{A}$ there exists a homeomorphism

$$\Phi'_\alpha\colon P'\big|_{U_\alpha} \longrightarrow U_\alpha \times G \qquad \text{s.t.}$$
$$\pi_{\alpha;1}\circ\Phi'_\alpha = \pi_{P'}, \quad \pi_{\alpha;2}\big(\Phi'_\alpha(p'u)\big) = \big(\pi_{\alpha;2}\big(\Phi'_\alpha(p')\big)\big)\cdot u \quad \forall\, p'\in P|_{U_\alpha},\ u\in G.$$

For every $\alpha\in\mathcal{A}$, there then exists a continuous map

$$f_\alpha\colon U_\alpha \longrightarrow G \qquad \text{s.t.} \quad \pi_{\alpha;2}\big(\Phi'_\alpha\big(\Psi(p)\big)\big) = f_\alpha\big(\pi_P(p)\big)\cdot\big(\pi_{\alpha;2}\big(\Phi_\alpha(p)\big)\big) \quad \forall\, p\in P\big|_{U_\alpha}.$$

The transition data $g_{P'} \equiv (g'_{\alpha\beta})_{\alpha,\beta\in\mathcal{A}}$ determined by the collection $\{\Phi'_\alpha\}_{\alpha\in\mathcal{A}}$ of trivializations of P' then satisfies

$$g'_{\alpha\beta} = f_\alpha\big|_{U_{\alpha\beta}}\cdot g_{\alpha\beta}\cdot f_\beta^{-1}\big|_{U_{\alpha\beta}} \qquad \forall\,\alpha,\beta\in\mathcal{A}.$$

Thus, $g_{P'} = f * g_P$, where $f \equiv (f_\alpha)_{\alpha\in\mathcal{A}}$, and

$$\big[g_{P'}\big] = \big[g_P\big] \in \check{H}^1(Y;G).$$

We conclude that the element $[g_P]\in\check{H}^1(Y;G)$ constructed above depends only on the isomorphism class of the principal G-bundle P over Y.

Conversely, suppose $[g]\in\check{H}^1(Y;G)$. Let $\underline{U} \equiv (U_\alpha)_{\alpha\in\mathcal{A}}$ be an open cover of Y and $g \equiv \{g_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}}$ be an element of $\ker\delta_1 \subset \check{C}^1(\underline{U};\mathcal{S}_Y(G))$ representing $[g]$. Define

$$\pi_{P_g}\colon P_g = \Big(\bigsqcup_{\alpha\in\mathcal{A}} \{\alpha\}\times U_\alpha \times G\Big)\Big/\!\sim_g \longrightarrow Y,$$
$$\big(\alpha, y, g_{\alpha\beta}(y)u\big) \sim_g \big(\beta, y, u\big) \quad \forall\,\alpha,\beta\in\mathcal{A},\ (y,u)\in U_\beta\times G\,.$$

This is a principal G-bundle over Y with trivializations

$$\Phi_\alpha\colon P_g\big|_{U_\alpha} \longrightarrow U_\alpha\times G, \quad \Phi_\alpha\big([\alpha,y,u)]\big) = (y,u),$$

for $\alpha\in\mathscr{A}$ and the associated transition data g. Thus,

$$\big[g_{P_g}\big] = [g] \in \check{H}^1(Y;G). \tag{A.38}$$

We show below that the isomorphism class $[P_g]$ of P_g depends only on $[g]$.

Suppose $\underline{U}'\equiv\{U'_\alpha\}_{\alpha\in\mathscr{A}'}$ is a refinement of $\underline{U}$ and $\mu\colon\mathscr{A}'\longrightarrow\mathscr{A}$ is a refining map. The map

$$\Psi\colon P_{\mu^*g}\equiv\Big(\bigsqcup_{\alpha\in\mathscr{A}'}\{\alpha\}\times U'_\alpha\times G\Big)\Big/\!\sim_{\mu^*g}\longrightarrow P_g\equiv\Big(\bigsqcup_{\alpha\in\mathscr{A}}\{\alpha\}\times U_\alpha\times G\Big)\Big/\!\sim_g,$$
$$\Psi\big([\alpha,y,u]\big) = \big[\mu(\alpha),y,u\big],$$

is then an isomorphism of principal G-bundles. Thus, it is sufficient to show that if

$$g,g'\in\ker\delta_1\subset\check{C}^1\big(\underline{U};\mathcal{S}_Y(G)\big) \qquad\text{and}\qquad [g]=[g']\in\check{H}^1\big(\underline{U};\mathcal{S}_Y(G)\big),$$

then the principal G-bundles P_g and $P_{g'}$ are isomorphic. By definition, there exists

$$f\equiv(f_\alpha)_{\alpha\in\mathscr{A}}\in\check{C}^0\big(\underline{U};\mathcal{S}_Y(G)\big) \qquad\text{s.t.}\qquad g'=f*g.$$

The map

$$\Psi\colon P_g=\Big(\bigsqcup_{\alpha\in\mathscr{A}}\{\alpha\}\times U_\alpha\times G\Big)\Big/\!\sim_g\longrightarrow P_{g'}=\Big(\bigsqcup_{\alpha\in\mathscr{A}}\{\alpha\}\times U_\alpha\times G\Big)\Big/\!\sim_{g'},$$
$$\Psi\big([\alpha,y,u]\big) = \big[\alpha,y,f_\alpha(y)\cdot u\big],$$

is then an isomorphism of principal G-bundles.

Let P be a principal G-bundle over Y, $\{\Phi_\alpha\}_{\alpha\in\mathscr{A}}$ be a collection of trivializations of P over an open cover $\underline{U}\equiv(U_\alpha)_{\alpha\in\mathscr{A}}$, and $g_P\equiv(g_{\alpha\beta})_{\alpha,\beta\in\mathscr{A}}$ be the corresponding transition data. The map

$$\Psi\colon P\longrightarrow P_{g_P}\equiv\Big(\bigsqcup_{\alpha\in\mathscr{A}}\{\alpha\}\times U_\alpha\times G\Big)\Big/\!\sim_g, \quad \Psi(p)=\big[\alpha,\Phi_\alpha(p)\big] \;\;\forall\,p\in P|_{U_\alpha},\,\alpha\in\mathscr{A},$$

is then an isomorphism of principal G-bundles. Along with (A.38), this implies that the maps

$$\begin{aligned} \mathrm{Prin}_Y(G)&\longrightarrow\check{H}^1(Y;G), &\qquad [P]&\longrightarrow[g_P],\\ \check{H}^1(Y;G)&\longrightarrow\mathrm{Prin}_Y(G), &\qquad [g]&\longrightarrow[P_g], \end{aligned} \tag{A.39}$$

are mutual inverses that identify $\mathrm{Prin}_Y(G)$ with $\check{H}^1(Y;G)$.

If $f\colon Y\longrightarrow Y'$ is a continuous map and $P\longrightarrow Y'$ is a principal G-bundle, then

$$\big[g_{f^*P}\big] = f^*\big[g_P\big]\in\check{H}^1(Y;G).$$

Thus, the identifications (A.39) are natural with respect to continuous maps.

Corollary A.7. *Let Y be a paracompact locally H^1-simple space and Φ_Y be as in (A.1). For every real line bundle V over Y,*

$$\check{H}^1\big(Y;\mathbb{Z}_2\big) \ni \Phi_Y\big(w_1(V)\big) = \big[g_{\mathrm{O}(V)}\big] \in \check{H}^1\big(Y;\mathrm{O}(1)\big)$$

under the canonical identification of the groups $\mathbb{Z}_2$ and $\mathrm{O}(1)$.

Proof. By the Universal Coefficient Theorem for Cohomology [26, Theorem 53.5], the homomorphism

$$\kappa\colon H^1(Y;\mathbb{Z}_2) \longrightarrow \mathrm{Hom}\big(\pi_1(Y), H^1(S^1;\mathbb{Z}_2)\big), \qquad \big\{\kappa(\eta)\big\}\big(f\colon S^1\longrightarrow Y\big) = f^*\eta,$$

is injective. By the naturality of w_1, Φ_Y, and (A.39), it is thus sufficient to show that

$$\check{H}^1\big(\mathbb{RP}^1;\mathbb{Z}_2\big) \ni \Phi_{\mathbb{RP}^1}\big(w_1(f^*V)\big) = \big[g_{\mathrm{O}(f^*V)}\big] \in \check{H}^1\big(\mathbb{RP}^1;\mathrm{O}(1)\big) \qquad \text{(A.40)}$$

for every continuous map $f\colon \mathbb{RP}^1\longrightarrow Y$. Since every line bundle over the interval $[0,1]$ is trivializable, the line bundle f^*V is isomorphic to either the trivial line bundle τ_1 or the real tautological line bundle $\gamma_{\mathbb{R};1}$. Both sides of (A.40) vanish in the first case. Since (A.39) is a bijection, this implies that the right-hand side of (A.40) does not vanish in the second case. The left-hand side of (A.40) does not vanish in this case by the *Normalization Axiom* for Stiefel-Whitney classes [24, p38]. □

A.5 Orientable vector bundle over surfaces

We now combine the description of complex line bundles in terms of Čech cohomology and the identification of some Čech cohomology groups with the singular ones to characterize orientable vector bundles over surfaces and their trivializations.

Lemma A.8. *Let Y be a paracompact locally contractible space. The homomorphism*

$$c_1\colon \mathrm{LB}_{\mathbb{C}}(Y) \longrightarrow H^2(Y;\mathbb{Z}), \qquad L \longrightarrow c_1(L),$$

from the group of isomorphism classes of complex line bundles is an isomorphism.

Proof. By Section A.4, there is a natural bijection

$$\mathrm{LB}_{\mathbb{C}}(Y) \longrightarrow \check{H}^1(Y;\mathbb{C}^*);$$

it is a group isomorphism in this case. By the proof of Proposition A.1, there are natural isomorphisms

$$\check{H}^p(Y;\mathbb{Z}) \approx H^p(Y;\mathbb{Z}) \qquad \forall\, p\in\mathbb{Z}.$$

By the reasoning in [32, Section 5.10], $\mathcal{S}_Y(\mathbb{C})$ is a fine sheaf. Along with [32, p202], this implies that

$$\check{H}^p\big(Y;\mathcal{S}_Y(\mathbb{C})\big) = \{0\} \qquad \forall\, p\in\mathbb{Z}^+.$$

Since Y is locally contractible, it is locally simple with respect to the short exact sequence

$$\{0\} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \longrightarrow \{0\}$$

of abelian Lie groups in the sense of Definition A.4. Thus, we obtain a commutative diagram

$$\begin{array}{ccccccc}
\{0\}=\check{H}^1(\mathbb{CP}^\infty;\mathbb{C}) & \longrightarrow & \mathrm{LB}_{\mathbb{C}}(\mathbb{CP}^\infty) & \xrightarrow{\check{\delta}_1} & H^2\big(\mathbb{CP}^\infty;\mathbb{Z}\big) & \longrightarrow & \check{H}^2(\mathbb{CP}^\infty;\mathbb{C})=\{0\} \\
\downarrow f^* & & \downarrow f^* & & \downarrow f^* & & \downarrow f^* \\
\{0\}=\check{H}^1(Y;\mathbb{C}) & \longrightarrow & \mathrm{LB}_{\mathbb{C}}(Y) & \xrightarrow{\check{\delta}_1} & H^2(Y;\mathbb{Z}) & \longrightarrow & \check{H}^2(Y;\mathbb{C})=\{0\}
\end{array} \tag{A.41}$$

of group homomorphisms for every continuous map $f\colon Y\longrightarrow\mathbb{CP}^\infty$.

Let $\gamma_{\mathbb{C}}\longrightarrow\mathbb{CP}^\infty$ be the complex tautological line bundle. By [24, Theorem 14.5], $H^2(\mathbb{CP}^\infty;\mathbb{Z})$ is freely generated by $c_1(\gamma_{\mathbb{C}})$. Along with the exactness of the top row in (A.41), this implies that $\check{\delta}_1=\pm c_1$ in this row. By [24, Theorem 14.6], for every complex line bundle L over Y there exists a continuous map $f\colon Y\longrightarrow\mathbb{CP}^\infty$ such that $L=f^*\gamma_{\mathbb{C}}$. Along with the commutativity of (A.41), these statements imply that $\check{\delta}_1=\pm c_1$ in the bottom row in (A.41) as well. The claim now follows from the exactness of this row. □

Remark A.9. The statement and proof of Lemma A.8 apply to any paracompact space Y satisfying the $k=2$ case of Definition 3.2 with $H^p(\cdot;\mathbb{Z}_2)$ replaced by $H^p(\cdot;\mathbb{Z})$.

Corollary A.10. *Let Y be a CW complex with cells of dimension at most 2. If $H^2(Y;\mathbb{Z})=\{0\}$, then every orientable vector bundle V over Y is trivializable.*

Proof. Let $n=\mathrm{rk}\,V$. If $n=1$, then V is an orientable line bundle and is thus trivializable. Suppose $n\geqslant2$. Since the cells of Y are of dimension at most 2, there exists a rank 2 orientable vector bundle L over Y such that

$$V\approx L\oplus\big(Y\times\mathbb{R}^{n-2}\big). \tag{A.42}$$

The real vector bundle L admits a complex structure $\mathfrak{i}$. It can be obtained by fixing an orientation and a metric on L and defining $\mathfrak{i}v\in L$ for $v\in L$ nonzero to be the vector which is orthogonal to v and has the same length as v so that $v,\mathfrak{i}v$ form an oriented basis for a fiber of L. By Lemma A.8, $(L,\mathfrak{i})$ is trivializable as a complex line bundle. Along with (A.42), this establishes the claim. □

Corollary A.11. *Let Σ be a surface, possibly with boundary, and $n\geqslant3$. The map*

$$\mathrm{OVB}_n(\Sigma)\longrightarrow H^2(\Sigma;\mathbb{Z}_2),\qquad V\longrightarrow w_2(V),$$

from the set of isomorphism classes of rank n oriented vector bundles over Σ is a bijection.

Proof. We can assume that Σ is connected. If Σ is not compact or has boundary, then

$$H^2(Y;\mathbb{Z}),H^2(\Sigma;\mathbb{Z}_2)=\{0\}.$$

By Corollary A.10, we can thus assume that Σ is closed and so $H^2(\Sigma;\mathbb{Z}_2)\approx\mathbb{Z}_2$.

Let $C\subset\Sigma$ be an embedded loop separating Σ into two surfaces, Σ_1 and Σ_2, with boundary C. By Corollary A.10, a rank n oriented vector bundle V over Σ is isomorphic to the vector bundle obtained by gluing $\Sigma_1\times\mathbb{R}^n$ and $\Sigma_2\times\mathbb{R}^n$ along $C\times\mathbb{R}^n$ by a clutching map $\varphi\colon C\longrightarrow\mathrm{SO}(n)$. Since $n\geqslant3$, $\pi_1(\mathrm{SO}(n))\approx\mathbb{Z}_2$. It thus remains to show that there exists an orientable vector bundle V

over Σ with $w_2(V)\neq 0$.

Let $\gamma_{\mathbb{C};1}\longrightarrow\mathbb{CP}^1$ be the complex tautological line bundle. Since $w_2(\gamma_{\mathbb{C};1})$ is the image of $c_1(\gamma_{\mathbb{C};1})$ under the reduction homomorphism

$$H^2\big(\mathbb{CP}^1;\mathbb{Z}\big)\longrightarrow H^2\big(\mathbb{CP}^1;\mathbb{Z}_2\big),$$

$w_2(\gamma_{\mathbb{C};1})\neq 0$ by the proof of Lemma A.8. If $f:\Sigma\longrightarrow\mathbb{CP}^1$ is a degree 1 map with respect to the $\mathbb{Z}_2$-coefficients, then

$$\big\langle w_2(f^*\gamma_{\mathbb{C};1}),[\Sigma]_{\mathbb{Z}_2}\big\rangle=\big\langle w_2(\gamma_{\mathbb{C};1}),f_*[\Sigma]_{\mathbb{Z}_2}\big\rangle=\big\langle w_2(\gamma_{\mathbb{C};1}),[\mathbb{CP}^1]_{\mathbb{Z}_2}\big\rangle\neq 0.$$

Thus, w_2 of the orientable vector bundle

$$V\equiv f^*\gamma_{\mathbb{C};1}\oplus\big(\Sigma\times\mathbb{R}^{n-2}\big)\longrightarrow\Sigma$$

is nonzero. □

Corollary A.12. *Suppose $\widetilde\Sigma$ is a compact surface with two boundary components and $\widehat\Sigma$ is a closed surface obtained from $\widetilde\Sigma$ by identifying these components with each other. Let $n\in\mathbb{Z}^+$ and $\widehat V\longrightarrow\widehat\Sigma$ be the orientable vector bundle obtained from $\widetilde\Sigma\times\mathbb{R}^n$ by identifying its restrictions to $\partial\widetilde\Sigma$ via a clutching map $\varphi\colon S^1\longrightarrow\mathrm{SO}(n)$. If $\widehat\Sigma$ is connected and $n\geqslant 3$, then φ is homotopically trivial if and only if $w_2(\widehat V)=0$.*

Proof. By Corollary A.10, every rank n orientable vector bundle over $\widetilde\Sigma$ is trivializable. Thus, every rank n orientable vector bundle $\widehat V$ over $\widehat\Sigma$ is obtained from $\widetilde\Sigma\times\mathbb{R}^n$ by identifying its restrictions to the two components of $\partial\widetilde\Sigma$ via a clutching map $\varphi\colon S^1\longrightarrow\mathrm{SO}(n)$. Since

$$\pi_1\big(\mathrm{SO}(n)\big)\approx\mathbb{Z}_2\qquad\text{and}\qquad H^2(\widehat\Sigma;\mathbb{Z}_2)\approx\mathbb{Z}_2,$$

the claim thus follows from Corollary A.11. □

Corollary A.13. *Let Σ be a compact connected surface with boundary components $C,C_1,\ldots,C_k$ and V be an orientable vector bundle over Σ. If $\mathrm{rk}\,V\geqslant 3$, then every trivialization of V over $C_1\cup\ldots\cup C_k$ extends to a trivialization Ψ of V over Σ and the homotopy class of the restriction of Ψ to $V|_C$ is determined by the homotopy class of its restriction to $V|_{C_1\cup\ldots\cup C_k}$.*

Proof. Let $n=\mathrm{rk}\,V$ and choose an orientation on V. Denote by $\widehat\Sigma$ the connected surface with one boundary component C obtained from Σ by attaching the 2-disks D_i^2 along the boundary components C_i. Let $\widehat V$ be the oriented vector bundle over $\widehat\Sigma$ obtained by identifying each $D_i^2\times\mathbb{R}^n$ with V over C_i via the chosen trivialization ϕ_i. By Corollary A.10, the oriented vector bundle $\widehat V$ admits a trivialization Ψ. Since there is a unique homotopy class of trivializations of $\widehat V|_{D_i^2}$, the restriction of Ψ to $V|_{C_i}$ is homotopic to ϕ_i and thus can deformed to be the same.

Suppose Ψ,Ψ' are trivializations of $V\longrightarrow\Sigma$ restricting to the same trivializations ϕ_i of $V|_{C_i}$ for every $i=1,\ldots,k$. Denote by $\widehat\Sigma$ (resp. $\widetilde\Sigma$) the closed (resp. compact) surface obtained from two copies of Σ, Σ and Σ', by identifying them along the boundary components corresponding to $C,C_1,\ldots,C_k$ (resp. $C_1,\ldots,C_k$). Thus, $\widetilde\Sigma$ has two boundary components, each of which corresponds to C, and $\widehat\Sigma$ can be obtained from $\widetilde\Sigma$ by identifying these two boundary components. Let

$$\widetilde q\colon\widetilde\Sigma\longrightarrow\Sigma\qquad\text{and}\qquad\widehat q\colon\widehat\Sigma\longrightarrow\Sigma$$

be the natural projections. The trivializations Ψ and Ψ' induce a trivialization $\widetilde{\Psi}$ of $\widetilde{q}^*V$ over $\widetilde{\Sigma}$ which restricts to Ψ and Ψ' over $\Sigma, \Sigma' \subset \widehat{\Sigma}, \widetilde{\Sigma}$. The bundle $\widehat{q}^*V$ over $\widehat{\Sigma}$ is obtained from $\widetilde{q}^*V$ by identifying the copies of $V|_C$ via the clutching map $\varphi\colon S^1 \longrightarrow \mathrm{SO}(n)$ determined by the difference between the trivializations of $V|_C$ induced by Ψ and Ψ'. Since

$$w_2\big(\widehat{q}^*V\big) = \widehat{q}^*w_2(V) = 0 \in H^2(\widehat{\Sigma};\mathbb{Z}_2),$$

Corollary A.12 implies that φ is homotopically trivial. Thus, Ψ and Ψ' determine the same homotopy class of trivializations of $V|_C$. $\square$

For an oriented vector bundle $V \longrightarrow Y$, let $\mathrm{Triv}(V)$ denote the set of homotopy classes of trivializations of V. For an oriented vector bundle V over a surface Σ, we define the map

$$\varepsilon_V\colon \mathrm{Triv}\big(V|_{\partial\Sigma}\big) \longrightarrow \mathbb{Z}_2 \tag{A.43}$$

by setting $\varepsilon_V(\phi)=0$ for the trivializations ϕ of $V|_{\partial\Sigma}$ that extend to trivializations of V over Σ and $\varepsilon_V(\phi)=1$ for the trivializations ϕ that do not.

Corollary A.14. *Let Σ be a compact connected surface with $\partial\Sigma \neq \varnothing$ and V be an oriented vector bundle over Σ. If $\mathrm{rk}\,V \geqslant 3$, then the map (A.43) is surjective and changing the homotopy class of a trivialization ϕ over precisely one component of $\partial\Sigma$ changes the value $\varepsilon_V(\phi)$.*

Proof. This follows from $\pi_1(\mathrm{SO}(n)) \approx \mathbb{Z}_2$ and Corollary A.13. $\square$

B Lie group covers

This appendix reviews basic statements concerning covers of Lie groups by Lie groups that are Lie group homomorphisms. Lemma B.1 describes the structure of connected Lie group covers. Lemma B.2 and Proposition B.3 do the same for covers of disconnected Lie groups with connected restrictions to the identity component of the base. We conclude with examples involving the groups $\mathrm{Spin}(n)$ and $\mathrm{Pin}^{\pm}(n)$ defined in Sections 2.1 and 2.2, respectively.

B.1 Terminology and summary

We call a covering projection $q\colon \widetilde{G} \longrightarrow G$ a Lie group covering if $\widetilde{G}$ and G are Lie groups and q is a Lie group homomorphism. We call such a cover connected if $\widetilde{G}$ is connected; this implies that so is G. Lie group coverings

$$q\colon \widetilde{G} \longrightarrow G \qquad \text{and} \qquad q'\colon \widetilde{G}' \longrightarrow G$$

are equivalent if there exists a Lie group isomorphism $\rho\colon \widetilde{G} \longrightarrow \widetilde{G}'$ such that $q = q' \circ \rho$. For a connected Lie group G, we denote by $\mathrm{Cov}(G)$ the set of equivalence classes of connected Lie group coverings of G and by $\pi_1(G)$ its fundamental group based at the identity $\mathbb{1}$. For any group H, we denote by $\mathrm{SG}(H)$ the set of subgroups of H. The next lemma is established in Section B.2.

Lemma B.1. *(a) Let G be a connected Lie group. The map*

$$\mathrm{Cov}(G) \longrightarrow \mathrm{SG}\big(\pi_1(G)\big), \qquad \big[q\colon \widetilde{G} \longrightarrow G\big] \longrightarrow q_*\pi_1\big(\widetilde{G}\big),$$

is a bijection. For every $[q] \in \mathrm{Cov}(G)$ as above, $q^{-1}(\mathbb{1})$ is contained in the center of $\widetilde{G}$.