Basic Riemannian Geometry and Sobolev Estimates used in Symplectic Topology

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Abstract

This note collects a number of standard statements in Riemannian geometry and in Sobolev-space theory that play a prominent role in analytic approaches to symplectic topology. These include relations between connections and complex structures, estimates on exponential-like maps, and dependence of constants in Sobolev and elliptic estimates.

Contents

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1 Connections in real vector bundles

1.1 Connections and splittings

Suppose M is a smooth manifold and $\pi_E\colon E \longrightarrow M$ is a vector bundle. We identify M with the zero section of E. Denote by

$$\mathfrak{a}\colon E\oplus E \longrightarrow E \qquad \text{and} \qquad \pi_{E\oplus E}\colon E\oplus E \longrightarrow M$$

the associated addition map and the induced projection map, respectively. For $f\in C^\infty(M;\mathbb{R})$, define

$$m_f\colon E \longrightarrow E \qquad \text{by} \qquad m_f(v) = f\big(\pi_E(v)\big)\cdot v \quad \forall\, v\in E. \tag{1.1}$$

In particular,

$$\pi_{E\oplus E} = \pi_E\circ\mathfrak{a}, \qquad \pi_E = \pi_E\circ m_f \quad \forall\, f\in C^\infty(M;\mathbb{R}).$$

The total spaces of the vector bundles

$$\pi_{E\oplus E}\colon E\oplus E \longrightarrow M \qquad \text{and} \qquad \pi_E^*E \longrightarrow E$$

consist of the pairs (v,w) in $E\times E$ such that $\pi_E(v)=\pi_E(w)$.

Define a smooth bundle homomorphism

$$\iota_E\colon \pi_E^*E \longrightarrow TE, \qquad \iota_E(v,w) = \frac{\mathrm{d}}{\mathrm{d}t}(v+tw)\Big|_{t=0}. \tag{1.2}$$

Since the restriction of ι_E to the fiber over $v\in E$ is the composition of the isomorphism

$$E_{\pi_E(v)} \longrightarrow T_vE_{\pi_E(v)}, \qquad w \longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}(v+tw)\Big|_{t=0},$$

with the differential of the embedding of the fiber $E_{\pi_E(v)}$ into E, ι_E is an injective bundle homomorphism. Furthermore,

$$\mathrm{d}\pi_E\circ\iota_E = 0, \quad m_f^*\iota_E\circ\pi_E^*m_f = \mathrm{d}m_f\circ\iota_E, \quad \mathfrak{a}^*\iota_E\circ\pi_{E\oplus E}^*\mathfrak{a} = \mathrm{d}\mathfrak{a}\circ\iota_{E\oplus E}, \tag{1.3}$$

$$TE|_M \approx TM\oplus \operatorname{Im}\iota_E\,. \tag{1.4}$$

By the first statement in (1.3), the injectivity of ι_E, and surjectivity of $\mathrm{d}\pi_E$,

$$0 \longrightarrow \pi_E^*E \xrightarrow{\iota_E} TE \xrightarrow{\mathrm{d}\pi_E} \pi_E^*TM \longrightarrow 0 \tag{1.5}$$

is an exact sequence of vector bundles over E. By the second statement in (1.3), the diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_E^*E \arrow[r, "\iota_E"] \arrow[d, "\pi_E^*m_f"] & TE \arrow[r, "\mathrm{d}\pi_E"] \arrow[d, "\mathrm{d}m_f"] & \pi_E^*TM \arrow[r] \arrow[d, "\pi_E^*\mathrm{id}"] & 0 \\
0 \arrow[r] & \pi_E^*E \arrow[r, "m_f^*\iota_E"] & m_f^*TE \arrow[r, "m_f^*\mathrm{d}\pi_E"] & \pi_E^*TM \arrow[r] & 0
\end{tikzcd} \tag{1.6}$$

of vector bundle homomorphisms over E commutes. By the third statement in (1.3), the diagram

$$\begin{tikzcd} 0 \arrow[r] & \pi^*_{E\oplus E}(E\oplus E) \arrow[r, "\iota_{E\oplus E}"] \arrow[d, "\pi^*_{E\oplus E}\mathfrak{a}"] & T(E\oplus E) \arrow[r, "\mathrm{d}\pi_{E\oplus E}"] \arrow[d, "\mathrm{d}\mathfrak{a}"] & \pi^*_{E\oplus E}TM \arrow[r] \arrow[d, "\pi^*_{E\oplus E}\mathrm{id}"] & 0 \\ 0 \arrow[r] & \pi^*_{E\oplus E}E \arrow[r, "\mathfrak{a}^*\iota_E"] & \mathfrak{a}^*TE \arrow[r, "\mathfrak{a}^*\mathrm{d}\pi_E"] & \pi^*_{E\oplus E}TM \arrow[r] & 0 \end{tikzcd} \tag{1.7}$$

of vector bundle homomorphisms over $E\oplus E$ commutes.

A connection in E is an $\mathbb{R}$-linear map

$$\begin{aligned} &\nabla\colon \Gamma(M;E) \longrightarrow \Gamma(M;T^*M\otimes_{\mathbb{R}}E) \qquad \text{s.t.} \\ &\nabla(f\xi) = \mathrm{d}f\otimes\xi + f\nabla\xi \quad \forall\ f\in C^\infty(M),\ \xi\in\Gamma(M;E). \end{aligned} \tag{1.8}$$

The Leibnitz property (1.8) implies that any two connections in E differ by a 1-form on M. In other words, if ∇ and $\widetilde{\nabla}$ are connections in E there exists

$$\begin{aligned} &\theta \in \Gamma\big(M;T^*M\otimes_{\mathbb{R}}\mathrm{Hom}_{\mathbb{R}}(E,E)\big) \qquad \text{s.t.} \\ &\widetilde{\nabla}_v\xi = \nabla_v\xi + \big\{\theta(v)\big\}\xi \qquad \forall\ \xi\in\Gamma(M;E),\ v\in T_xM,\ x\in M. \end{aligned} \tag{1.9}$$

If U is a neighborhood of $x\in M$ and f is a smooth function on M supported in U such that $f(x)=1$, then

$$\nabla\xi\big|_x = \nabla\big(f\xi\big)\big|_x - \mathrm{d}_xf\otimes\xi(x) \tag{1.10}$$

by (1.8). The right-hand side of (1.10) depends only on $\xi|_{\mathrm{U}}$. Thus, a connection ∇ in E is a local operator, i.e. the value of $\nabla\xi$ at a point $x\in M$ depends only on the restriction of ξ to any neighborhood U of x.

Suppose U is an open subset of M and $\xi_1,\ldots,\xi_n\in\Gamma(\mathrm{U};E)$ is a frame for E on U, i.e.

$$\xi_1(x),\ldots,\xi_n(x)\in E_x$$

is a basis for E_x for all $x\in\mathrm{U}$. By definition of ∇, there exist

$$\theta_l^k\in\Gamma(U;T^*U) \qquad \text{s.t.} \qquad \nabla\xi_l = \sum_{k=1}^{k=n}\xi_k\theta_l^k \equiv \sum_{k=1}^{k=n}\theta_l^k\otimes\xi_k \quad \forall\ l=1,\ldots,n.$$

We call

$$\theta\equiv\big(\theta_l^k\big)_{k,l=1,\ldots,n}\in\Gamma\big(U;T^*U\otimes_{\mathbb{R}}\mathrm{Mat}_n\mathbb{R}\big)$$

the connection 1-form of ∇ with respect to the frame $(\xi_k)_k$.

For an arbitrary section

$$\xi=\sum_{l=1}^{l=n}f^l\xi_l\in\Gamma(\mathrm{U};E),$$

by (1.8) we have

$$\nabla\xi=\sum_{k=1}^{k=n}\xi_k\Big(\mathrm{d}f^k+\sum_{l=1}^{l=n}\theta_l^k f^l\Big),\qquad \text{i.e.}\qquad \nabla\big(\underline{\xi}\cdot\underline{f}^t\big)=\underline{\xi}\cdot\big\{\mathrm{d}+\theta\big\}\underline{f}^t, \tag{1.11}$$

$$\text{where}\qquad \underline{\xi}=(\xi_1,\ldots,\xi_n),\quad \underline{f}=(f^1,\ldots,f^n). \tag{1.12}$$

This implies that

$$\nabla\xi\big|_x=\pi_2|_x\circ\mathrm{d}_x\xi\colon T_xM\longrightarrow E_x\qquad \forall\,\xi\in\Gamma(U;E)\text{ s.t. }\xi(x)=0, \tag{1.13}$$

where $\pi_2|_x\colon T_xE\longrightarrow E_x$ is the projection to the second component in (1.4).

By (1.11), ∇ is a first-order differential operator. By (1.8), its symbol is given by

$$\sigma_\nabla\colon T^*M\longrightarrow \mathrm{Hom}\big(E,T^*M\otimes_{\mathbb{R}}E\big),\qquad \big\{\sigma_\nabla(\eta)\big\}(f)=\eta\otimes f.$$

Lemma 1.1. *Suppose M is a smooth manifold and $\pi_E\colon E\longrightarrow M$ is a vector bundle. A connection ∇ in E induces a splitting*

$$TE\approx\pi_E^*TM\oplus\pi_E^*E \tag{1.14}$$

of the exact sequence (1.5) extending the splitting (1.4) such that

$$\nabla\xi\big|_x=\pi_2|_x\circ\mathrm{d}_x\xi\colon T_xM\longrightarrow E_x\qquad \forall\;\xi\in\Gamma(M;E),\;x\in M, \tag{1.15}$$

where $\pi_2|_x\colon T_xE\longrightarrow E_x$ is the projection onto the second component in (1.14). Furthermore,

$$\mathrm{d}m_t\approx\pi_E^*\mathrm{id}\oplus\pi_E^*m_t\;\;\forall\;t\in\mathbb{R}\qquad \textit{and}\qquad \mathfrak{a}\approx\pi_{E\oplus E}^*\mathrm{id}\oplus\pi_{E\oplus E}^*\mathfrak{a}, \tag{1.16}$$

with respect to the splitting (1.14), i.e. it is consistent with the commutative diagrams (1.6) and (1.7).

Proof. Given $x\in M$ and $v\in E_x$, choose $\xi\in\Gamma(M;E)$ such that $\xi(x)=v$ and let

$$T_vE^{\mathrm{h}}=\mathrm{Im}\,\{\mathrm{d}\xi-\nabla\xi\}\big|_x\subset T_vE.$$

Since $\pi_E\circ\xi=\mathrm{id}_M$,

$$\mathrm{d}_v\pi_E\circ\big\{\mathrm{d}\xi-\nabla\xi\big\}\big|_x=\mathrm{id}_{T_xM}\qquad\Longrightarrow\qquad T_vE\approx T_vE^{\mathrm{h}}\oplus E_x\approx T_xM\oplus E_x.$$

This splitting of T_vE satisfies (1.15) at v.

With the notation as in (1.11),

$$\{\mathrm{d}\xi-\nabla\xi\}\big|_x=\Big(\mathrm{d}_x\mathrm{id}_M,\sum_{l=1}^{l=n}f^l(x)\theta_l^1|_x,\ldots\sum_{l=1}^{l=n}f^l(x)\theta_l^n|_x\Big)\colon T_xM\longrightarrow T_xM\oplus\mathbb{R}^n$$

with respect to the identification $E|_U\approx U\times\mathbb{R}^k$ determined by the frame $(\xi_k)_k$. Thus, T_vE^{h} is independent of the choice of ξ. Furthermore, the resulting splitting (1.14) of (1.5) extends (1.4) and satisfies (1.16). $\square$

1.2 Metric-compatible connections

Suppose $E \longrightarrow M$ is a smooth vector bundle. Let g be a metric on E, i.e.

$$g \in \Gamma(M; E^* \otimes_{\mathbb{R}} E^*) \qquad \text{s.t.} \qquad g(v,w) = g(w,v), \quad g(v,v) > 0 \quad \forall\, v, w \in E_x,\ v \neq 0,\ x \in M.$$

A connection ∇ in E is g-compatible if

$$\mathrm{d}\big(g(\xi,\zeta)\big) = g(\nabla\xi,\zeta) + g(\xi,\nabla\zeta) \in \Gamma(M; T^*M) \qquad \forall\, \xi,\zeta \in \Gamma(M;E).$$

Suppose U is an open subset of M and $\xi_1,\ldots,\xi_n \in \Gamma(\mathrm{U};E)$ is a frame for E on U. For $i,j=1,\ldots,n$, let

$$g_{ij} = g(\xi_i,\xi_j) \in C^\infty(\mathrm{U}).$$

If ∇ is a connection in E and θ_{kl} is the connection 1-form for ∇ with respect to the frame $\{\xi_k\}_k$, then ∇ is g-compatible on U if and only if

$$\sum_{k=1}^{k=n} \big(g_{ik}\theta_j^k + g_{jk}\theta_i^k\big) = \mathrm{d}g_{ij} \qquad \forall\, i,j = 1,2,\ldots,n. \tag{1.17}$$

1.3 Torsion-free connections

If M is a smooth manifold, a connection ∇ in TM is torsion-free if

$$\nabla_X Y - \nabla_Y X = [X,Y].$$

If $(x_1,\ldots,x_n)\colon \mathrm{U} \longrightarrow \mathbb{R}^n$ is a coordinate chart on M, let

$$\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n} \in \Gamma(\mathrm{U};TM)$$

be the corresponding frame for TM on U. If ∇ is a connection in TM, the corresponding connection 1-form θ can be written as

$$\theta_j^k = \sum_{i=1}^{i=n} \Gamma_{ij}^k \mathrm{d}x^i, \qquad \text{where} \qquad \nabla_{\partial/\partial x_i}\frac{\partial}{\partial x_j} = \sum_{k=1}^{k=n} \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

The connection ∇ is torsion-free on $TM|_{\mathrm{U}}$ if and only if

$$\Gamma_{ij}^k = \Gamma_{ji}^k \qquad \forall\, i,j,k = 1,\ldots,n. \tag{1.18}$$

Lemma 1.2. *If (M,g) is a Riemannian manifold, there exists a unique torsion-free g-compatible connection ∇ in TM.*

Proof. (1) Suppose ∇ and $\widetilde{\nabla}$ are torsion-free g-compatible connections in TM. By (1.9), there exists

$$\begin{gathered}\theta\in\Gamma\big(M;T^*M\otimes_{\mathbb{R}}\mathrm{Hom}_{\mathbb{R}}(TM,TM)\big)\qquad\text{s.t.}\\ \widetilde{\nabla}_XY-\nabla_XY=\big\{\theta(X)\big\}Y\qquad\forall\,Y\in\Gamma(M;TM),\ X\in T_xM,\ x\in M.\end{gathered}$$

Since ∇ and $\widetilde{\nabla}$ are torsion-free,

$$\big\{\theta(X)\big\}Y=\big\{\theta(Y)\big\}X\qquad\forall\,X,Y\in T_xM,\ x\in M. \tag{1.19}$$

Since ∇ and $\widetilde{\nabla}$ are g-compatible,

$$\begin{cases}g\big(\{\theta(X)\}Y,Z\big)+g\big(Y,\{\theta(X)\}Z\big)=0\\ g\big(\{\theta(Y)\}X,Z\big)+g\big(X,\{\theta(Y)\}Z\big)=0\\ g\big(\{\theta(Z)\}X,Y\big)+g\big(X,\{\theta(Z)\}Y\big)=0\end{cases}\qquad\forall\,X,Y,Z\in T_xM,\ x\in M. \tag{1.20}$$

Adding the first two equations in (1.20), subtracting the third, and using (1.19) and the symmetry of g, we obtain

$$2g\big(\{\theta(X)\}Y,Z\big)=0\quad\forall\,X,Y,Z\in T_xM,\ x\in M\qquad\Longrightarrow\qquad\theta\equiv0.$$

Thus, $\widetilde{\nabla}=\nabla$.

(2) Let $(x_1,\ldots,x_n)\colon \mathrm{U}\longrightarrow\mathbb{R}^n$ be a coordinate chart on M. With notation as in the paragraph preceding Lemma 1.2, ∇ is g-compatible on $TM|_{\mathrm{U}}$ if and only if

$$\sum_{l=1}^{l=n}\big(g_{il}\Gamma_{kj}^l+g_{jl}\Gamma_{ki}^l\big)=\partial_{x_k}g_{ij}; \tag{1.21}$$

see (1.17). Define a connection ∇ in $TM|_{\mathrm{U}}$ by

$$\Gamma_{ij}^k=\frac{1}{2}\sum_{l=1}^{l=n}g^{kl}\big(\partial_{x_i}g_{jl}+\partial_{x_j}g_{il}-\partial_{x_l}g_{ij}\big)\qquad\forall\,i,j,k=1,\ldots,n,$$

where g^{ij} is the (i,j)-entry of the inverse of the matrix $(g_{ij})_{i,j=1,\ldots,n}$. Since $g_{ij}=g_{ji}$, Γ_{ij}^k satisfies (1.18); a direct computation shows that Γ_{ij}^k also satisfies (1.21). Therefore, ∇ is a torsion-free g-compatible connection on $TM|_{\mathrm{U}}$. In this way, we can define a torsion-free g-compatible connection on every coordinate chart. By the uniqueness property, these connections agree on the overlaps. $\square$

2 Complex structures

2.1 Complex linear connections

Suppose M is a smooth manifold and $\pi\colon(E,\mathfrak{i})\longrightarrow M$ is a complex vector bundle. Similarly to Section 1.1, there is an exact sequence

$$0\longrightarrow\pi_E^*E\xrightarrow{\iota_E}TE\xrightarrow{\mathrm{d}\pi_E}\pi_E^*TM\longrightarrow0 \tag{2.1}$$

of vector bundles over E. The homomorphism ι_E is now $\mathbb{C}$-linear. If $f\in C^\infty(M;\mathbb{C})$ and $m_f\colon E\longrightarrow E$ is defined as in (1.1), there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_E^*E \arrow[r, "\iota_E"] \arrow[d, "\pi_E^*m_f"] & TE \arrow[r, "\mathrm{d}\pi_E"] \arrow[d, "\mathrm{d}m_f"] & \pi_E^*TM \arrow[r] \arrow[d, "\pi_E^*\mathrm{id}"] & 0 \\
0 \arrow[r] & \pi_E^*E \arrow[r, "m_f^*\iota_E"] & m_f^*TE \arrow[r, "m_f^*\mathrm{d}\pi_E"] & \pi_E^*TM \arrow[r] & 0
\end{tikzcd} \tag{2.2}$$

of bundle maps over E.

Suppose

$$\nabla\colon\Gamma(M;E)\longrightarrow\Gamma(M;T^*M\otimes_{\mathbb{R}}E)$$

is a $\mathbb{C}$-linear connection, i.e.

$$\nabla_v(\mathfrak{i}\xi)=\mathfrak{i}(\nabla_v\xi)\qquad\forall\,\xi\in\Gamma(M;E),\,v\in TM.$$

If U is an open subset of M and $\xi_1,\ldots,\xi_n\in\Gamma(\mathrm{U};E)$ is a $\mathbb{C}$-frame for E on U, then there exist

$$\theta_l^k\in\Gamma(M;T^*M)\qquad\text{s.t.}\qquad\nabla\xi_l=\sum_{k=1}^{k=n}\xi_k\theta_l^k\equiv\sum_{k=1}^{k=n}\theta_l^k\otimes\xi_k\quad\forall\,l=1,\ldots,n.$$

We will call

$$\theta\equiv\big(\theta_l^k\big)_{k,l=1,\ldots,n}\in\Gamma\big(\Sigma;T^*M\otimes_{\mathbb{R}}\mathrm{Mat}_n\mathbb{C}\big)$$

the complex connection 1-form of ∇ with respect to the frame $(\xi_k)_k$. For an arbitrary section

$$\xi=\sum_{l=1}^{l=n}f^l\xi_l\in\Gamma(\mathrm{U};E),$$

by (1.8) and $\mathbb{C}$-linearity of ∇ we have

$$\nabla\xi=\sum_{k=1}^{k=n}\xi_k\Big(\mathrm{d}f^k+\sum_{l=1}^{l=n}\theta_l^kf^l\Big),\qquad\text{i.e.}\qquad\nabla\big(\underline{\xi}\cdot\underline{f}^t\big)=\underline{\xi}\cdot\big\{\mathrm{d}+\theta\big\}\underline{f}^t,\tag{2.3}$$

where $\underline{\xi}$ and $\underline{f}$ are as (1.12).

Let g be a hermitian metric on E, i.e.

$$g\in\Gamma\big(M;\mathrm{Hom}_{\mathbb{C}}(\bar{E}\otimes_{\mathbb{C}}E,\mathbb{C})\big)\quad\text{s.t.}\quad g(v,w)=\overline{g(w,v)},\quad g(v,v)>0\quad\forall\,v,w\in E_x,\ v\neq0,\ x\in M.$$

A $\mathbb{C}$-linear connection ∇ in E is g-compatible if

$$\mathrm{d}\big(g(\xi,\zeta)\big)=g(\nabla\xi,\zeta)+g(\xi,\nabla\zeta)\in\Gamma(M;T^*M\otimes_{\mathbb{R}}\mathbb{C})\qquad\forall\,\xi,\zeta\in\Gamma(M;E).$$

With notation as in the previous paragraph, let

$$g_{ij}=g(\xi_i,\xi_j)\in C^\infty(\mathrm{U};\mathbb{C})\qquad\forall\,i,j=1,\ldots,n.$$

Then ∇ is g-compatible on U if and only if

$$\sum_{k=1}^{k=n}\big(g_{ik}\theta_j^k+\bar{g}_{jk}\bar{\theta}_i^k\big)=\mathrm{d}g_{ij}\qquad\forall\,i,j=1,2,\ldots,n.\tag{2.4}$$

2.2 Generalized $\bar{\partial}$-operators

If $(\Sigma,\mathfrak{j})$ is an almost complex manifold, let

$$T^*\Sigma^{1,0}\equiv\big\{\eta\in T^*\Sigma\otimes_{\mathbb{R}}\mathbb{C}\colon \eta\circ\mathfrak{j}=\mathfrak{i}\,\eta\big\}\qquad\text{and}\qquad T^*\Sigma^{0,1}\equiv\big\{\eta\in T^*\Sigma\otimes_{\mathbb{R}}\mathbb{C}\colon \eta\circ\mathfrak{j}=-\mathfrak{i}\,\eta\big\}$$

be the bundles of $\mathbb{C}$-linear and $\mathbb{C}$-antilinear 1-forms on Σ. If $(\Sigma,\mathfrak{j})$ and (M,J) are smooth almost complex manifolds and $u\colon\Sigma\longrightarrow M$ is a smooth function, define

$$\bar{\partial}_{J,\mathfrak{j}}u\in\Gamma\big(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}u^*TM\big)\qquad\text{by}\qquad \bar{\partial}_{J,\mathfrak{j}}u=\frac{1}{2}\big(\mathrm{d}u+J\circ\mathrm{d}u\circ\mathfrak{j}\big). \tag{2.5}$$

A smooth map $u\colon(\Sigma,\mathfrak{j})\longrightarrow(M,J)$ will be called $(J,\mathfrak{j})$-holomorphic if $\bar{\partial}_{J,\mathfrak{j}}u=0$.

Definition 2.1. *Suppose $(\Sigma,\mathfrak{j})$ is an almost complex manifold and $\pi:(E,\mathfrak{i})\longrightarrow\Sigma$ is a complex vector bundle. A $\bar{\partial}$-operator on $(E,\mathfrak{i})$ is a $\mathbb{C}$-linear map*

$$\bar{\partial}\colon\Gamma(\Sigma;E)\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E)$$

such that

$$\bar{\partial}\big(f\xi\big)=(\bar{\partial}f)\otimes\xi+f(\bar{\partial}\xi)\qquad\forall\ f\in C^\infty(\Sigma),\ \xi\in\Gamma(\Sigma;E), \tag{2.6}$$

where $\bar{\partial}f=\bar{\partial}_{\mathfrak{i},\mathfrak{j}}f$ is the usual $\bar{\partial}$-operator on complex-valued functions.

Similarly to Section 1.1, a $\bar{\partial}$-operator on $(E,\mathfrak{i})$ is a first-order differential operator. If U is an open subset of M and $\xi_1,\ldots,\xi_n\in\Gamma(\mathrm{U};E)$ is a $\mathbb{C}$-frame for E on U, then there exist

$$\theta_l^k\in\Gamma(U;T^*U^{0,1})\qquad\text{s.t.}\qquad \bar{\partial}\xi_l=\sum_{k=1}^{k=n}\xi_k\theta_l^k\equiv\sum_{k=1}^{k=n}\theta_l^k\otimes\xi_k\quad\forall\ l=1,\ldots,n.$$

We call

$$\theta\equiv\big(\theta_l^k\big)_{k,l=1,\ldots,n}\in\Gamma\big(U;T^*U^{0,1}\otimes_{\mathbb{C}}\mathrm{Mat}_n\mathbb{C}\big)$$

the connection 1-form of $\bar{\partial}$ with respect to the frame $(\xi_k)_k$. For an arbitrary section

$$\xi=\sum_{l=1}^{l=n}f^l\xi_l\in\Gamma(\mathrm{U};E),$$

by (2.6) we have

$$\bar{\partial}\xi=\sum_{k=1}^{k=n}\xi_k\Big(\bar{\partial}f^k+\sum_{l=1}^{l=n}\theta_l^kf^l\Big),\qquad\text{i.e.}\qquad \bar{\partial}\big(\underline{\xi}\cdot\underline{f}^t\big)=\underline{\xi}\cdot\{\bar{\partial}+\theta\}\underline{f}^t, \tag{2.7}$$

where $\underline{\xi}$ and $\underline{f}$ are as in (1.12). It is immediate from (2.6) that the symbol of $\bar{\partial}$ is given by

$$\sigma_{\bar{\partial}}\colon T^*\Sigma\longrightarrow\mathrm{Hom}_{\mathbb{C}}\big(E,T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E\big),\qquad \big\{\sigma_{\bar{\partial}}(\eta)\big\}(f)=\big(\eta+\mathfrak{i}\,\eta\circ\mathfrak{j}\big)\otimes f.$$

In particular, $\bar{\partial}$ is an elliptic operator (i.e. $\sigma_{\bar{\partial}}(\eta)$ is an isomorphism for $\eta\neq0$) if $(\Sigma,\mathfrak{j})$ is a Riemann surface.

Lemma 2.2. *Suppose* $(\Sigma,\mathfrak{j})$ *is an almost complex manifold and* $\pi\colon(E,\mathfrak{i})\longrightarrow\Sigma$ *is a complex vector bundle. If*

$$\bar\partial\colon\Gamma(\Sigma;E)\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E)$$

is a $\bar\partial$*-operator on* $(E,\mathfrak{i})$*, there exists a unique almost complex structure* $J=J_{\bar\partial}$ *on (the total space of)* E *such that* π *is a* $(\mathfrak{j},J)$*-holomorphic map, the restriction of* J *to the vertical tangent bundle* $TE^{\mathrm{v}}\approx\pi^*E$ *agrees with* $\mathfrak{i}$*, and*

$$\bar\partial_{J,\mathfrak{j}}\xi=0\in\Gamma(\mathrm{U};T^*\Sigma^{0,1}\otimes_{\mathbb{C}}\xi^*TE)\qquad\Longleftrightarrow\qquad\bar\partial\xi=0\in\Gamma(\mathrm{U};T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E)\tag{2.8}$$

for every open subset U *of* Σ *and* $\xi\in\Gamma(\mathrm{U};E)$*.*

Proof. (1) With notation as above, define

$$\varphi\colon\mathrm{U}\times\mathbb{C}^n\longrightarrow E|_{\mathrm{U}}\qquad\text{by}\qquad\varphi(x,c^1,\ldots,c^n)=\underline{\xi}(x)\cdot\underline{c}^t\equiv\sum_{k=1}^{k=n}c^k\xi_k(x)\in E_x.$$

The map φ is a trivialization of E over U. If $J\equiv J_{\bar\partial}$ is an almost complex structure on E, let $\widetilde{J}$ be the almost complex structure on $\mathrm{U}\times\mathbb{C}^n$ given by

$$\widetilde{J}_{(x,\underline{c})}=\big\{\mathrm{d}_{(x,\underline{c})}\varphi\big\}^{-1}\circ J_{\varphi(x,\underline{c})}\circ\mathrm{d}_{(x,\underline{c})}\varphi\qquad\forall\ (x,\underline{c})\in\mathrm{U}\times\mathbb{C}^n.\tag{2.9}$$

The almost complex structure J restricts to $\mathfrak{i}$ on TE^{v} if and only if

$$\widetilde{J}_{(x,\underline{c})}w=\mathfrak{i}w\in T_{\underline{c}}\mathbb{C}^n\subset T_{(x,\underline{c})}(\mathrm{U}\times\mathbb{C}^n)\qquad\forall\ w\in T_{\underline{c}}\mathbb{C}^n.\tag{2.10}$$

If J restricts to $\mathfrak{i}$ on TE^{v}, the projection π is $(\mathfrak{j},J)$-holomorphic on $E|_{\mathrm{U}}$ if and only if there exists

$$\begin{gathered}\widetilde{J}^{\mathrm{vh}}\in\Gamma\big(\mathrm{U}\times\mathbb{C}^n;\mathrm{Hom}_{\mathbb{R}}(\pi_{\mathrm{U}}^*T\mathrm{U},\pi_{\mathbb{C}^n}^*T\mathbb{C}^n)\big)\qquad\text{s.t.}\\ \widetilde{J}_{(x,\underline{c})}w=\mathfrak{j}_xw+\widetilde{J}^{\mathrm{vh}}_{(x,\underline{c})}w\qquad\forall\ w\in T_x\mathrm{U}\subset T_{(x,\underline{c})}(\mathrm{U}\times\mathbb{C}^n).\end{gathered}\tag{2.11}$$

If $\xi\in\Gamma(\mathrm{U};E)$, let

$$\widetilde{\xi}\equiv\varphi^{-1}\circ\xi\equiv\big(\mathrm{id}_{\mathrm{U}},\underline{f}\big),\qquad\text{where}\qquad\underline{f}\in C^\infty(\mathrm{U};\mathbb{C}^n).$$

By (2.9)-(2.11),

$$\begin{split}2\,\bar\partial_{J,\mathfrak{j}}\xi\big|_x=\mathrm{d}_{\widetilde{\xi}(x)}\varphi\circ2\bar\partial_{\widetilde{J},\mathfrak{j}}\widetilde{\xi}\big|_x&=\mathrm{d}_{\widetilde{\xi}(x)}\varphi\circ\big\{\big(\mathrm{Id}_{T_x\mathrm{U}},\mathrm{d}_x\underline{f}\big)+\widetilde{J}_{\widetilde{\xi}(x)}\circ\big(\mathrm{Id}_{T_x\mathrm{U}},\mathrm{d}_x\underline{f}\big)\circ\mathfrak{j}_x\big\}\\&=\mathrm{d}_{\widetilde{\xi}(x)}\varphi\circ\big(0,2\,\bar\partial f|_x+\widetilde{J}^{\mathrm{vh}}_{\widetilde{\xi}(x)}\circ\mathfrak{j}_x\big).\end{split}\tag{2.12}$$

On the other hand, by (2.7),

$$\begin{split}\bar\partial\xi|_x=\bar\partial\big(\underline{\xi}\cdot f^t\big)\big|_x&=\underline{\xi}(x)\cdot\big\{\bar\partial+\theta\big\}f^t\big|_x\\&=\varphi\big(\bar\partial f|_x+\theta_x\cdot f(x)^t\big).\end{split}\tag{2.13}$$

By (2.12) and (2.13), the property (2.8) is satisfied for all $\xi\in\Gamma(\mathrm{U};E)$ if and only if

$$\widetilde{J}^{\mathrm{vh}}_{(x,\underline{c})}=2\big(\theta_x\cdot\underline{c}^t\big)\circ(-\mathfrak{j}_x)=2\mathfrak{i}\,\theta_x\cdot\underline{c}^t\qquad\forall\ (x,\underline{c})\in\mathrm{U}\times\mathbb{C}^n.$$

In summary, the almost complex structure $J\!=\!J_{\bar\partial}$ on E has the three desired properties if and only if for every trivialization of E over an open subset U of Σ

$$\begin{gathered}\widetilde{J}_{(x,\underline{c})}\big(w_1,w_2\big)=\big(\mathfrak{j}_xw_1,\mathfrak{i}w_2+2\mathfrak{i}\theta_x(w_1)\cdot\underline{c}^t\big)\\ \forall\ (x,\underline{c})\in\mathrm{U}\!\times\!\mathbb{C}^n,\ (w_1,w_2)\in T_x\mathrm{U}\!\oplus\!T_{\underline{c}}\mathbb{C}^n=T_{(x,\underline{c})}(\mathrm{U}\!\times\!\mathbb{C}^n),\end{gathered}\tag{2.14}$$

where $\widetilde{J}$ is the almost complex structure on $\mathrm{U}\!\times\!\mathbb{C}^n$ induced by J via the trivialization and θ is the connection 1-form corresponding to $\bar\partial$ with respect to the frame inducing the trivialization.

(2) By (2.14), there exists at most one almost complex structure J satisfying the three properties. Conversely, (2.14) determines such an almost complex structure on E. Since

$$\begin{split}\widetilde{J}^2_{(x,\underline{c})}\big(w_1,w_2\big)=\widetilde{J}_{(x,\underline{c})}\big(\mathfrak{j}w_1,\mathfrak{i}w_2+2\mathfrak{i}\theta_x(w_1)\cdot\underline{c}^t\big)&=\big(\mathfrak{j}^2w_1,\mathfrak{i}\big(\mathfrak{i}w_2+2\mathfrak{i}\theta_x(w_1)\cdot\underline{c}^t\big)+2\mathfrak{i}\theta_x(\mathfrak{j}w_1)\cdot\underline{c}^t\big)\\ &=-(w_1,w_2),\end{split}$$

$\widetilde{J}$ is indeed an almost complex structure on E. The almost complex structure induced by $\widetilde{J}$ on $E|_{\mathrm{U}}$ satisfies the three properties by part (a). By the uniqueness property, the almost complex structures on E induced by the different trivializations agree on the overlaps. Therefore, they define an almost complex structure $J\!=\!J_{\bar\partial}$ on the total space of E with the desired properties. $\square$

2.3 Connections and $\bar\partial$-operators

Suppose $(\Sigma,\mathfrak{j})$ is an almost complex manifold, $\pi\colon(E,\mathfrak{i})\longrightarrow\Sigma$ is a complex vector bundle, and

$$\bar\partial\colon\Gamma(\Sigma;E)\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E)$$

is a $\bar\partial$-operator on $(E,\mathfrak{i})$. A $\mathbb{C}$-linear connection ∇ in $(E,\mathfrak{i})$ is $\bar\partial$-compatible if

$$\bar\partial\xi=\bar\partial_\nabla\xi\equiv\frac{1}{2}\big(\nabla\xi+\mathfrak{i}\nabla\xi\circ\mathfrak{j}\big)\qquad\forall\ \xi\!\in\!\Gamma(M;\Sigma).\tag{2.15}$$

Lemma 2.3. *Suppose $(\Sigma,\mathfrak{j})$ is an almost complex manifold, $\pi:(E,\mathfrak{i})\longrightarrow\Sigma$ is a complex vector bundle,*

$$\bar\partial\colon\Gamma(\Sigma;E)\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E)$$

is a $\bar\partial$-operator on $(E,\mathfrak{i})$, and $J_{\bar\partial}$ is the complex structure in the vector bundle $TE\longrightarrow E$ provided by Lemma 2.2. A $\mathbb{C}$-linear connection ∇ in $(E,\mathfrak{i})$ is $\bar\partial$-compatible if and only if the splitting (1.14) determined by ∇ respects the complex structures.

Proof. Since $J_{\bar\partial}=\pi^*\mathfrak{i}$ on $\pi^*E\subset TE$, the splitting (1.14) determined by ∇ respects the complex structures if and only if

$$J_{\bar\partial}|_v\circ\big\{\mathrm{d}\xi-\nabla\xi\big\}\big|_x=\big\{\mathrm{d}\xi-\nabla\xi\big\}\big|_x\circ\mathfrak{j}_x\colon T_x\Sigma\longrightarrow T_vE$$

for all $x\!\in\!\Sigma$, $v\!\in\!E_x$, and $\xi\!\in\!\Gamma(\Sigma;E)$ such that $\xi(x)\!=\!0$; see the proof of Lemma 1.1. This identity is equivalent to

$$\bar\partial_{J_{\bar\partial},\mathfrak{j}}\xi=\bar\partial_\nabla\xi\qquad\forall\ \xi\in\Gamma(\Sigma;E).\tag{2.16}$$

On the other hand, by the proof of Lemma 2.2,

$$\bar\partial_{J_{\bar\partial},\mathfrak{j}}\xi=\bar\partial\xi\qquad\forall\ \xi\in\Gamma(\Sigma;E);\tag{2.17}$$

see (2.12)-(2.14). The lemma follows immediately from (2.16) and (2.17). $\square$

2.4 Holomorphic vector bundles

Let $(\Sigma,\mathfrak{j})$ be a complex manifold. A holomorphic vector bundle $(E,\mathfrak{i})$ on $(\Sigma,\mathfrak{j})$ is a complex vector bundle with a collection of trivializations that overlap holomorphically.

A collection of holomorphically overlapping trivializations of (E,i) determines a holomorphic structure J on the total space of E and a $\bar{\partial}$-operator

$$\bar{\partial}\colon \Gamma(\Sigma;E) \longrightarrow \Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E).$$

The latter is defined as follows. If $\xi_1,\ldots,\xi_n$ is a holomorphic complex frame for E over an open subset U of M, then

$$\bar{\partial}\sum_{k=1}^{k=n} f^k\xi_k = \sum_{k=1}^{k=n}\bar{\partial}f^k\otimes\xi_k \qquad \forall\ f^1,\ldots,f^k\in C^\infty(\mathrm{U};\mathbb{C}).$$

In particular, for all $\xi\in\Gamma(M;E)$

$$\bar{\partial}_{J,\mathfrak{j}}\xi=0 \qquad\Longleftrightarrow\qquad \bar{\partial}\xi=0.$$

Thus, $J=J_{\bar{\partial}}$; see Lemma 2.2.

Lemma 2.4. *Suppose $(\Sigma,\mathfrak{j})$ is a Riemann surface and $\pi\colon(E,\mathfrak{i})\longrightarrow\Sigma$ is a complex vector bundle. If*

$$\bar{\partial}\colon \Gamma(\Sigma;E) \longrightarrow \Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E)$$

is a $\bar{\partial}$-operator on $(E,\mathfrak{i})$, the almost complex structure $J=J_{\bar{\partial}}$ on E is integrable. With this complex structure, $\pi\colon E\longrightarrow\Sigma$ is a holomorphic vector bundle and $\bar{\partial}$ is the corresponding $\bar{\partial}$-operator.

Proof. By (2.8), it is sufficient to show that there exists a $(J,\mathfrak{j})$-holomorphic local section through every point $v\in E$, i.e. there exist a neighborhood U of $x\equiv\pi(v)$ in Σ and $\xi\in\Gamma(\mathrm{U};E)$ such that

$$\xi(x)=v \qquad\text{and}\qquad \bar{\partial}_{J,\mathfrak{j}}\xi=0.$$

By Lemma 2.2 and (2.13), this is equivalent to showing that the equation

$$\Big\{\bar{\partial}+\theta\Big\}f^t=0,\qquad f(x)=v,\qquad f\in C^\infty(\mathrm{U};\mathbb{C}^n), \tag{2.18}$$

has a solution for every $v\in\mathbb{C}^n$. We can assume that U is a small disk contained in S^2. Let

$$\eta\colon S^2\longrightarrow[0,1]$$

be a smooth function supported in U and such that $\eta\equiv1$ on a neighborhood of x. Then,

$$\eta\theta\in\Gamma(S^2;(T^*S^2)^{0,1}\otimes_{\mathbb{C}}\mathrm{Mat}_n\mathbb{C}).$$

Choose $p>2$. The operator

$$\Theta: L^p_1(S^2;\mathbb{C}^n)\longrightarrow L^p\big(S^2;(T^*S^2)^{0,1}\otimes_{\mathbb{C}}\mathbb{C}^n\big)\oplus\mathbb{C}^n,\qquad \Theta(f)=\big(\bar{\partial}_{\mathfrak{i},\mathfrak{j}}f,f(x)\big),$$

is surjective. If η has sufficiently small support, so is the operator

$$\Theta_\eta: L^p_1(S^2;\mathbb{C}^n)\longrightarrow L^p\big(S^2;(T^*S^2)^{0,1}\otimes_{\mathbb{C}}\mathbb{C}^n\big)\oplus\mathbb{C}^n,\qquad \Theta_\eta(f)=\big(\{\bar{\partial}_{\mathfrak{i},\mathfrak{j}}+\eta\theta\}f,f(x)\big).$$

Then, the restriction of $\Theta_\eta^{-1}(0,v)$ to a neighborhood of x on which $\eta\equiv1$ is a solution of (2.18). By elliptic regularity, $\Theta_\eta^{-1}(0,v)\in C^\infty(S^2;\mathbb{C}^n)$. $\square$

2.5 Deformations of almost complex submanifolds

If (M,J) is a complex manifold, holomorphic coordinate charts on (M,J) determine a holomorphic structure in the vector bundle $(TM,\mathfrak{i})\longrightarrow M$. If $(\Sigma,\mathfrak{j})\subset(M,J)$ is a complex submanifold, holomorphic coordinate charts on Σ can be extended to holomorphic coordinate charts on M. Thus, the holomorphic structure in $T\Sigma\longrightarrow\Sigma$ induced from $(\Sigma,\mathfrak{j})$ is the restriction of the holomorphic structure in $TM|_\Sigma$. It follows that

$$\bar\partial_M=\bar\partial_\Sigma\colon\Gamma(\Sigma;T\Sigma)\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}T\Sigma)\subset\Gamma\big(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}TM|_\Sigma\big),$$

where $\bar\partial_M$ and $\bar\partial_\Sigma$ are the $\bar\partial$-operators in $TM|_\Sigma$ and $T\Sigma$ induced from the holomorphic structures in Σ and M. Therefore, $\bar\partial_M$ descends to a $\bar\partial$-operator on the quotient

$$\bar\partial\colon\Gamma(\Sigma;\mathcal{N}_M\Sigma)=\Gamma(\Sigma;TM|_\Sigma)/\Gamma(\Sigma;T\Sigma)\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}\mathcal{N}_M\Sigma),$$

where

$$\mathcal{N}_M\Sigma\equiv TM|_\Sigma/T\Sigma\longrightarrow\Sigma$$

is the normal bundle of Σ in M. This vector bundle inherits a holomorphic structure from that of $TM|_\Sigma$ and Σ. The above $\bar\partial$-operator on $\mathcal{N}_M$ is the $\bar\partial$-operator corresponding to this induced holomorphic structure on $\mathcal{N}_M\Sigma$.

Suppose (M,J) is an almost complex manifold and $(\Sigma,\mathfrak{j})\subset(M,J)$ is an almost complex submanifold. Let ∇ be a torsion-free connection in TM. Define

$$\begin{aligned}D_{J;\Sigma}\colon\Gamma(\Sigma;TM|_\Sigma)&\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}TM|_\Sigma)\qquad\text{by}\\ D_{J;\Sigma}\xi&=\frac12\big(\nabla\xi+J\circ\nabla\xi\circ\mathfrak{j}\big)-\frac12J\circ\nabla_\xi J\colon T\Sigma\longrightarrow TM|_\Sigma.\end{aligned}\tag{2.19}$$

If ∇ is the Levi-Civita connection (the connection of Lemma 1.2) for a J-compatible metric on M (and Σ is a Riemann surface), then $D_{J;\Sigma}$ is the linearization of the $\bar\partial_J$-operator at the inclusion map $\iota\colon\Sigma\longrightarrow M$; see [4, Proposition 3.1.1].

In fact, $D_{J;\Sigma}$ is independent of the choice of a torsion-free connection in TM. Let

$$\widetilde\nabla=\nabla+\theta,\qquad\theta\in\Gamma\big(M;T^*M\otimes_{\mathbb{R}}\mathrm{Hom}_{\mathbb{R}}(TM,TM)\big),\tag{2.20}$$

be another torsion-free connection; see (1.9). Since $\widetilde\nabla$ and ∇ are torsion-free connections,

$$\big\{\theta(X)\big\}Y=\big\{\theta(Y)\big\}X\qquad\forall X,Y\in T_xM,\ x\in M.\tag{2.21}$$

If $x\in M$ and $X,Y\in\Gamma(M;TM)$,

$$\begin{aligned}&\big\{\nabla_YJ\big\}X=\nabla_Y(JX)-J\nabla_YX\,,\quad\big\{\widetilde\nabla_YJ\big\}X=\widetilde\nabla_Y(JX)-J\widetilde\nabla_YX\qquad\Longrightarrow\\ &\big\{\widetilde\nabla_YJ\big\}X-\big\{\nabla_YJ\big\}X=\big\{\theta(Y)\big\}(JX)-J\big\{\theta(Y)\big\}X=\big\{\theta(JX)\big\}Y-J\big\{\theta(X)\big\}Y\end{aligned}\tag{2.22}$$

by (2.20) and (2.21). On the other hand, by (2.20) for all $X\in T\Sigma$ and $\xi\in\Gamma(\Sigma;TM|_\Sigma)$,

$$\begin{aligned}\big\{\widetilde\nabla\xi+J\circ\widetilde\nabla\xi\circ\mathfrak{j}\big\}(X)-\big\{\nabla\xi+J\circ\nabla\xi\circ\mathfrak{j}\big\}(X)&=\big\{\theta(X)\big\}\xi+J\big\{\theta(\mathfrak{j}X)\big\}\xi\\ &=J\big(\big\{\theta(JX)\big\}\xi-J\big\{\theta(X)\big\}\xi\big),\end{aligned}\tag{2.23}$$

since $\mathfrak{j}=J|_{T\Sigma}$ and $J^2=-\mathrm{Id}$. By (2.22) and (2.23), $D_{J,\Sigma}$ is independent of the choice of torsion-free connection ∇.

Since any torsion-free connection on Σ extends to a torsion-free connection on M, the above observation implies that

$$D_{J;\Sigma}\colon \Gamma(\Sigma;T\Sigma)\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}T\Sigma)\subset\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}TM|_{\Sigma}). \tag{2.24}$$

Thus, an almost complex submanifold $(\Sigma,\mathfrak{j})$ of an almost complex manifold (M,J) induces a well-defined generalized Cauchy-Riemann operator[1] on the normal bundle of Σ in M,

$$D^{\mathcal{N}}_{J;\Sigma}\colon \Gamma(\Sigma;\mathcal{N}_M\Sigma)\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}\mathcal{N}_M\Sigma),\qquad D^{\mathcal{N}}_{J;\Sigma}\big(\pi(\xi)\big)=\pi\big(D_{J;\Sigma}(\xi)\big)\ \ \forall\,\xi\in\Gamma(\Sigma;TM|_{\Sigma}),$$

where $\pi\colon TM|_{\Sigma}\longrightarrow\mathcal{N}_M\Sigma$ is the quotient projection map. The $\mathbb{C}$-linear part of $D^{\mathcal{N}}_{J;\Sigma}$ determines a $\bar\partial$-operator on the normal bundle of Σ in M:

$$\begin{gathered}\bar\partial^{\mathcal{N}}_{J;\Sigma}\colon \Gamma(\Sigma;\mathcal{N}_M\Sigma)\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}\mathcal{N}_M\Sigma),\\ \bar\partial^{\mathcal{N}}_{J;\Sigma}(\xi)=\frac12\big(D^{\mathcal{N}}_{J;\Sigma}(\xi)-JD^{\mathcal{N}}_{J;\Sigma}(J\xi)\big)\ \ \forall\,\xi\in\Gamma(\Sigma;\mathcal{N}_M\Sigma).\end{gathered}$$

Both operators are determined by the almost complex submanifold $(\Sigma,\mathfrak{j})$ of the almost complex manifold (M,J) only and are independent of the choice of torsion-free connection ∇ in (2.19).

Any connection ∇ in TM induces a J-linear connection in TM by

$$\nabla^J_X\xi=\nabla_X\xi-\frac12 J(\nabla_XJ)\xi\qquad\forall\,X\in TM,\ \xi\in\Gamma(M;TM). \tag{2.25}$$

If ∇ is as in (2.19),

$$\big\{D_{J;\Sigma}\xi\big\}(X)=\big\{\bar\partial_{\nabla^J}\xi\big\}(X)+A_J(X,\xi)-\frac14\big\{(\nabla_{J\xi}J)+J(\nabla_\xi J)\big\}(X) \tag{2.26}$$

for all $\xi\in\Gamma(\Sigma;TM|_{\Sigma})$ and $X\in T\Sigma$, where A_J is the Nijenhuis tensor of J:

$$A_J(\xi_1,\xi_2)=\frac14\Big([\xi_1,\xi_2]+J[\xi_1,J\xi_2]+J[J\xi_1,\xi_2]-[J\xi_1,J\xi_2]\Big)\qquad\forall\ \xi_1,\xi_2\in\Gamma(M;TM). \tag{2.27}$$

Since the sum of the terms in the curly brackets in (2.26) is $\mathbb{C}$-linear in ξ, while the Nijenhuis tensor is $\mathbb{C}$-antilinear, the $\mathbb{C}$-linear operator

$$\Gamma(\Sigma;TM|_{\Sigma})\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}TM|_{\Sigma}),\qquad \xi\longrightarrow\bar\partial_{\nabla^J}(\xi)-\frac14\big\{(\nabla_{J\xi}J)+J(\nabla_\xi J)\big\}, \tag{2.28}$$

takes $\Gamma(\Sigma;T\Sigma)$ to $\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}T\Sigma)$ by (2.24). Thus, it induces a $\bar\partial$-operator on $\mathcal{N}_M\Sigma$ and this induced operator is $\bar\partial^{\mathcal{N}}_{J;\Sigma}$. If the image of the homomorphism

$$TM\longrightarrow T^*\Sigma^{0,1}\otimes_{\mathbb{C}}TM|_{\Sigma}\,,\qquad \xi\longrightarrow\nabla_\xi J-J\nabla_{J\xi}J\,,$$

[1] see Section 4.3

is contained in $T^*\Sigma^{0,1}\otimes_{\mathbb{C}}T\Sigma$, then $\bar\partial_{\nabla^J}$ preserves $T\Sigma$ and induces a $\bar\partial$-operator $\bar\partial^{\mathcal{N}}_{\nabla^J}$ on $\mathcal{N}_M\Sigma$ with $\bar\partial^{\mathcal{N}}_{\nabla^J}=\bar\partial^{\mathcal{N}}_{J;\Sigma}$. In this case,

$$D^{\mathcal{N}}_{J;\Sigma}\big(\pi(\xi)\big)=\pi\big(\bar\partial_{\nabla^J}\xi+A_J(\cdot,\xi)\big)\colon T\Sigma\longrightarrow\mathcal{N}_M\Sigma \qquad \forall\,\xi\in\Gamma(\Sigma;TM|_\Sigma).$$

This is the case in particular if J is compatible with a symplectic form ω on M and ∇ is the Levi-Civita connection for the metric $g(\cdot,\cdot)=\omega(\cdot,J\cdot)$, as the sum in the curly brackets in (2.26) then vanishes by [4, (C.7.5)].

It is immediate that A_J takes $T\Sigma\otimes_{\mathbb{R}}T\Sigma$ to $T\Sigma$ and thus induces a bundle homomorphism

$$A^{\mathcal{N}}_J\colon T\Sigma\otimes_{\mathbb{R}}\mathcal{N}_M\Sigma\longrightarrow\mathcal{N}_M\Sigma\,.$$

If ζ is any vector field on M such that $\zeta(x)=X\in T_x\Sigma$ for some $x\in\Sigma$, then

$$\begin{gathered}\big\{D_{J;\Sigma}\xi\big\}(X)=\frac12\big([\zeta,\xi]+J[J\zeta,\xi]\big)\big|_x,\\ \Big\{\bar\partial_{\nabla^J}(\xi)-\frac14\big((\nabla_{J\xi}J)+J(\nabla_\xi J)\big)\Big\}(X)=\frac14\big([\zeta,\xi]+J[J\zeta,\xi]-J[\zeta,J\xi]+[J\zeta,J\xi]\big)\big|_x,\end{gathered}\tag{2.29}$$

since ∇ is torsion-free.[2] These two identities immediately imply that the operators (2.19) and (2.28) preserve $T\Sigma\subset TM|_\Sigma$ and thus induce operators

$$\Gamma(\Sigma;\mathcal{N}_M\Sigma)\longrightarrow\Gamma(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}\mathcal{N}_M\Sigma)$$

as claimed above.

If g is a J-compatible metric on $TM|_\Sigma$ and $\pi^\perp\colon TM|_\Sigma\longrightarrow T\Sigma^\perp$ is the projection to the g-orthogonal complement of $T\Sigma$ in $TM|_\Sigma$, the composition $\nabla^\perp$

$$\Gamma(\Sigma;T\Sigma^\perp)\hookrightarrow\Gamma\big(\Sigma;TM|_\Sigma\big)\xrightarrow{\nabla^J}\Gamma\big(\Sigma;T^*\Sigma\otimes_{\mathbb{R}}TM|_\Sigma\big)\xrightarrow{\pi^\perp}\Gamma\big(\Sigma;T^*\Sigma\otimes_{\mathbb{R}}T\Sigma^\perp\big),$$

with ∇^J as in (2.25), is a g-compatible J-linear connection in $T\Sigma^\perp$. Via the isomorphism $\pi:T\Sigma^\perp\longrightarrow\mathcal{N}_M\Sigma$, it induces a J-linear connection $\nabla^{\mathcal{N}}$ in $\mathcal{N}_M\Sigma$ which is compatible with the metric $g^{\mathcal{N}}$ induced via this isomorphism from $g|_{T\Sigma^\perp}$. If the image of the homomorphism

$$T\Sigma^\perp\longrightarrow T^*\Sigma^{0,1}\otimes_{\mathbb{C}}TM|_\Sigma\,,\qquad \xi\longrightarrow\nabla_\xi J-J\nabla_{J\xi}J\,,\tag{2.30}$$

is contained in $T^*\Sigma^{0,1}\otimes_{\mathbb{C}}T\Sigma$, then $\bar\partial_{\nabla^{\mathcal{N}}}=\bar\partial^{\mathcal{N}}_{J;\Sigma}$ and so

$$D^{\mathcal{N}}_{J;\Sigma}\big(\pi(\xi)\big)=\pi\big(\bar\partial_{\nabla^\perp}\xi+A_J(\cdot,\xi)\big)\colon T\Sigma\longrightarrow\mathcal{N}_M\Sigma\qquad\forall\,\xi\in\Gamma(\Sigma;T\Sigma^\perp).$$

This is the case if Σ is a divisor in M, i.e. $\mathrm{rk}_{\mathbb{C}}\mathcal{N}=1$, since $(\nabla_\zeta J)\xi$ is g-orthogonal to ξ and $J\xi$ for all $\xi,\zeta\in T_xM$ and $x\in M$ by [4, (C.7.1)]. This is also the case if J is compatible with a symplectic form ω on M and $g(\cdot,\cdot)=\omega(\cdot,J\cdot)$, as the homomorphism (2.30) is then trivial by [4, (C.7.5)].

[2]Since LHS and RHS of these identities depend only ξ and $X=\zeta(x)$, and not on ζ, it is sufficient to verify them under the assumption that $\nabla\zeta|_x=0$.

3 Riemannian geometry estimates

This section is based on [1, Chapter 1] and [2, Section 3] and culminates in a Poincare lemma for closed curves in Proposition 3.6 and an expansion for the $\bar{\partial}$-operator in Proposition 3.13. If $u\colon \Sigma \longrightarrow M$ is a smooth map between smooth manifolds and $E \longrightarrow M$ is a smooth vector bundle, let

$$\Gamma(u;E)=\Gamma(\Sigma;u^*E), \qquad \Gamma^1(u;E)=\Gamma(\Sigma;T^*\Sigma\otimes_{\mathbb{R}}u^*E).$$

We denote the subspace of compactly supported sections in $\Gamma(u;E)$ by $\Gamma_c(u;E)$.

An exponential-like map on a smooth manifold M is a smooth map $\exp: TM \longrightarrow M$ such that $\exp|_M = \mathrm{id}_M$ and

$$\mathrm{d}_x \exp = \big(\mathrm{id}_{T_xM}\ \mathrm{id}_{T_xM}\big)\colon T_x(TM)=T_xM\oplus T_xM \longrightarrow T_xM \qquad \forall\ x\in M,$$

where the second equality is the canonical splitting of $T_x(TM)$ into the horizontal and vertical tangent space along the zero section. Any connection ∇ in TM gives rise to a smooth map $\exp^\nabla: W \longrightarrow M$ from some neighborhood W of the zero section M in TM; see [1, Section 1.3]. If $\eta\colon TM \longrightarrow \mathbb{R}$ is a smooth function which equals 1 on a neighborhood of M in TM and 0 outside of W, then

$$\exp\colon TM \longrightarrow M, \qquad v \longrightarrow \exp^\nabla\big(\eta(v)v\big),$$

is an exponential-like map. If M is compact, then W can be taken to be all of TM and $\exp=\exp^\nabla$.

If $(M,g,\exp)$ is a Riemannian manifold with an exponential-like map and $x\in M$, let $r_{\exp}(x)\in\mathbb{R}^+$ be the supremum of the numbers $r\in\mathbb{R}$ such that the restriction

$$\exp\colon \big\{v\in T_xM\colon\ |v|<r\big\} \longrightarrow M$$

is a diffeomorphism onto an open subset of M. Set

$$r^g_{\exp}(x)=\inf\big\{d_g(x,\exp(v))\colon\ v\in T_xM,\ |v|=r_{\exp}(x)\big\}\in\mathbb{R}^+,$$

where d_g is the metric on M induced by g. If $K\subset M$, let

$$r^g_{\exp}(K)=\inf_{x\in K} r^g_{\exp}(x);$$

this number is positive if $\bar{K}\subset M$ is compact.

3.1 Parallel transport

Let $(E,\langle,\rangle,\nabla)\longrightarrow M$ be a vector bundle, real or complex, with an inner-product $\langle,\rangle$ and a metric-compatible connection ∇. If $\alpha\colon (a,b)\longrightarrow M$ is a piecewise smooth curve, denote by

$$\Pi_\alpha\colon E_{\alpha(a)} \longrightarrow E_{\alpha(b)}$$

the parallel-transport map along α with respect to the connection ∇. If $\exp: TM \longrightarrow M$ is an exponential-like map, $x\in M$, and $v\in T_xM$, let

$$\Pi_v\colon E_x \longrightarrow E_{\exp(v)}$$

be the parallel transport along the curve

$$\gamma_v\colon [0,1] \longrightarrow M, \qquad \gamma_v(t) = \exp(tv).$$

If $u\colon [a,b]\times[c,d] \longrightarrow M$ is a smooth map, let

$$\Pi_{\partial u} : E_{u(a,c)} \longrightarrow E_{u(a,c)}$$

be the parallel transport along u restricted to the boundary of the rectangle traversed in the positive direction. If $u\colon \Sigma \longrightarrow M$ is any smooth map, ∇ induces a connection

$$\nabla^u\colon \Gamma(u;E) \longrightarrow \Gamma^1(u;E)$$

in the vector bundle $u^*E \longrightarrow \Sigma$. If α is a smooth curve as above and $\zeta \in \Gamma(\alpha;E)$, let

$$\frac{D}{\mathrm{d}t}\zeta = \nabla^{\alpha}_{\partial_t}\zeta \in \Gamma(\alpha;E),$$

where ∂_t is the standard unit vector field on $\mathbb{R}$.

Lemma 3.1. *If (M,g) is a Riemannian manifold and $(E,\langle,\rangle,\nabla)$ is a normed vector bundle with connection over M, for every compact subset $K\subset M$ there exists $C_K\in\mathbb{R}^+$ such that for every smooth map $u\colon [a,b]\times[c,d] \longrightarrow M$ with $\mathrm{Im}\, u \subset K$*

$$|\Pi_{\partial u} - \mathbb{I}| \le C_K \int_c^d\int_a^b |u_s||u_t|\mathrm{d}s\mathrm{d}t,$$

where the norm of $(\Pi_{\partial u}-\mathbb{I})\in \mathrm{End}(E_{u(a,c)})$ is computed with respect to the inner-product in $E_{u(a,c)}$.

Proof. (1) Choose an orthonormal frame $\{v_i\}$ for $E_{u(a,c)}$. Extend each v_i to

$$\xi_i \in \Gamma\big(u|_{a\times[c,d]};E\big)$$

by parallel-transporting along the curve $t \longrightarrow u(a,t)$ and then to $\zeta_i \in \Gamma(u;E)$ by parallel-transporting $\xi_i(a,t)$ along the curve $s \longrightarrow u(s,t)$; see Figure 1. By construction,

$$\frac{D}{\mathrm{d}s}\zeta_i = 0 \in \Gamma(u;E).$$

Let A be the matrix-valued function on $[a,b]\times[c,d]$ such that

$$\frac{D}{\mathrm{d}t}\zeta_i\Big|_{(s,t)} = \sum_{l=1}^{l=k} A_{il}(s,t)\zeta_l(s,t), \tag{3.1}$$

where k is the rank of E. Note that $A_{ij}(a,t)=0$ and

$$\big\langle \mathcal{R}_\nabla(u_s,u_t)\zeta_i,\zeta_j\big\rangle = \Big\langle \frac{D}{\mathrm{d}s}\frac{D}{\mathrm{d}t}\zeta_i - \frac{D}{\mathrm{d}t}\frac{D}{\mathrm{d}s}\zeta_i, \zeta_j\Big\rangle = \sum_{l=1}^{l=k}\Big\langle\Big(\frac{\partial}{\partial s}A_{il}\Big)\zeta_l,\zeta_j\Big\rangle = \frac{\partial}{\partial s}A_{ij}\,, \tag{3.2}$$

where $\mathcal{R}_\nabla$ is the curvature tensor of the connection of ∇. Since K is compact and the image of u is contained in K, it follows that

$$|A_{ij}(b,t)| \le C_K\int_a^b |u_s|_{(s,t)}|u_t|_{(s,t)}\mathrm{d}s. \tag{3.3}$$

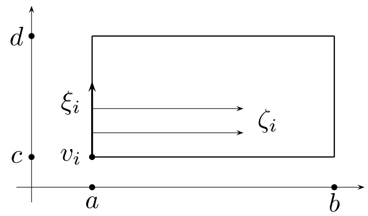

Figure 1: Extending a basis $\{v_i\}$ for $E_{u(a,c)}$ to a frame $\{\zeta_i\}$ over $[a,b]\times[c,d]$

(2) The parallel transport of ζ_i along the curves

$$\tau \longrightarrow u(\tau,c), \quad \tau \longrightarrow u(\tau,d), \quad \tau \longrightarrow u(a,\tau)$$

is ζ_i itself. Thus, it remains to estimate the parallel transport of each ζ_i along the curve $\tau \longrightarrow u(b,\tau)$. Let h_{ij} be the SO_k-valued function (U_k-valued function if E is complex) on $[c,d]$ such that

$$h(c)=\mathbb{I}, \qquad \sum_{j=1}^{j=k} \frac{D}{\mathrm{d}t}(h_{ij}\zeta_j)\Big|_{(b,t)} = 0 \;\; \forall\, i,t.$$

The second equation is equivalent to

$$\sum_{j=1}^{j=k} h'_{ij}(t)\zeta_j(b,t) + \sum_{j=1}^{j=k}\sum_{l=1}^{l=k} h_{ij}(t)A_{jl}(b,t)\zeta_l(b,t) = 0 \qquad \Longleftrightarrow \qquad h' = -hA(b,\cdot). \tag{3.4}$$

Since (the real part of) the trace of (A_{ij}) is zero by (3.2), equation (3.4) has a unique solution in SO_k (or U_k) such that $h(c)=\mathbb{I}$. Furthermore, by (3.3)

$$\big|h(d)-\mathbb{I}\big| \le \int_c^d |h'(t)|\mathrm{d}t \le \int_c^d |h||A|\mathrm{d}t \le k^2 \int_c^d\!\!\int_a^b C_K|u_s||u_t|\mathrm{d}s\mathrm{d}t. \tag{3.5}$$

Since $\Pi_{\partial\alpha}v_i = \sum_{j=1}^{j=k} h_{ij}(d)v_j$ by the above, the claim follows from equation (3.5). $\square$

Corollary 3.2. *If (M,g) is a Riemannian manifold and $(E,\langle,\rangle,\nabla)$ is a normed vector bundle with connection over M, for every compact subset $K\subset M$ there exists $C_K \in \mathbb{R}^+$ such that for every smooth closed curve $\alpha\colon [a,b]\longrightarrow M$ with $\operatorname{Im}\alpha \subset K$*

$$\big|\Pi_\alpha - \mathbb{I}\big| \le C_K \min\big(\|\mathrm{d}\alpha\|_1, (b-a)\|\mathrm{d}\alpha\|_2^2\big).$$

Proof. Let $\exp: TM \longrightarrow M$ be an exponential-like map. Since the group SO_k (or U_k if E is complex) is compact and

$$\|\mathrm{d}\alpha\|_1^2 \le (b-a)\|\mathrm{d}\alpha\|_2^2$$

by Hölder's inequality, it is enough to assume that

$$\|\mathrm{d}\alpha\|_1 \le \min(r^g_{\exp}(K)/2, 1).$$

Thus, there exists

$$\widetilde{\alpha}\in C^{\infty}\big([a,b];T_{\alpha(a)}M\big) \qquad \text{s.t.} \qquad \alpha(t)=\exp(\widetilde{\alpha}(t)), \quad |\widetilde{\alpha}(t)|_{\alpha(a)}<r_{\exp}(\alpha(a)).$$

Define

$$u\colon [0,1]\times[a,b]\longrightarrow K\subset M \qquad \text{by} \quad u(s,t)=\exp\big(s\widetilde{\alpha}(t)\big).$$

Using

$$\begin{aligned}|\widetilde{\alpha}(t)| &\le C_K d_g\big(\alpha(a),\alpha(t)\big)\le C_K\|\mathrm{d}\alpha\|_1\,,\\ |\widetilde{\alpha}'(t)| &= \big|\{\mathrm{d}_{\widetilde{\alpha}(t)}\exp\}^{-1}(\alpha'(t))\big|\le C_K|\mathrm{d}_t\alpha|\,,\end{aligned}$$

we find that

$$u_s(s,t)=\big\{\mathrm{d}_{s\widetilde{\alpha}(t)}\exp\big\}\big(\widetilde{\alpha}(t)\big) \qquad \Longrightarrow \qquad |u_s|_{(s,t)}\le C_K'\|\mathrm{d}\alpha\|_1\,; \tag{3.6}$$

$$u_t(s,t)=s\big\{\mathrm{d}_{s\widetilde{\alpha}(t)}\exp\big\}\big(\widetilde{\alpha}'(t)\big) \qquad \Longrightarrow \qquad |u_t|_{(s,t)}\le C_K'|\mathrm{d}_t\alpha|. \tag{3.7}$$

Thus, by Lemma 3.1,

$$\big|\Pi_\alpha-\mathbb{I}\big|=\big|\Pi_{\partial u}-\mathbb{I}\big|\le C_K\int_0^1\!\!\int_a^b|u_s||u_t|\mathrm{d}s\mathrm{d}t\le C_K'\|\mathrm{d}\alpha\|_1^2\le C_K'(b-a)\|\mathrm{d}\alpha\|_2^2.$$

Since $\|\mathrm{d}\alpha\|_1\le r^g_{\exp}(K)$, it follows that $|\Pi_\alpha-\mathbb{I}|\le C_K\|\mathrm{d}\alpha\|_1$. $\square$

Corollary 3.3. *If $(M,g,\exp)$ is a Riemannian manifold with an exponential-like map and $(E,\langle,\rangle,\nabla)$ is a normed vector bundle with connection over M, for every compact subset $K\subset M$ there exists $C_K\in C^{\infty}(\mathbb{R};\mathbb{R})$ such that for all $x\in K$ and smooth maps $\widetilde{\alpha}\colon(-\epsilon,\epsilon)\longrightarrow T_xM$ and $\xi\colon(-\epsilon,\epsilon)\longrightarrow E_x$*

$$\left|\frac{D}{\mathrm{d}t}\Big(\Pi_{\widetilde{\alpha}(t)}\xi(t)\Big)\Big|_{t=0}-\Pi_{\widetilde{\alpha}(0)}\xi'(0)\right|\le C_K\big(|\widetilde{\alpha}(0)|\big)|\widetilde{\alpha}(0)||\widetilde{\alpha}'(0)||\xi(0)|. \tag{3.8}$$

Proof. Define

$$u\colon[0,1]\times\big[0,\epsilon/2\big]\longrightarrow K\subset M \qquad \text{by} \quad u(s,t)=\exp\big(s\widetilde{\alpha}(t)\big).$$

Let $\{v_i\}$ be an orthonormal basis for E_x. Extend each v_i to

$$\zeta_i\in\Gamma\big(u|_{[0,1]\times t};E\big)$$

by parallel-transporting along the curves $s\longrightarrow u(s,t)$. If

$$\xi(t)=\sum_{i=1}^{i=k}f_i(t)v_i\,,$$

where k is the rank of E, then

$$\Pi_{\widetilde{\alpha}(t)}\xi(t)=\sum_{i=1}^{i=k}f_i(t)\zeta_i(1,t) \qquad \Longrightarrow$$

$$\begin{aligned}\frac{D}{\mathrm{d}t}\Big(\Pi_{\widetilde{\alpha}(t)}\xi(t)\Big)\Big|_{t=0}&=\sum_{i=1}^{i=k}f_i'(0)\zeta_i(1,0)+\sum_{i=1}^{i=k}f_i(0)\frac{D}{\mathrm{d}t}\zeta_i(1,t)\Big|_{t=0}\\ &=\Pi_{\widetilde{\alpha}(0)}\xi'(0)+\sum_{i=1}^{i=k}f_i(0)\frac{D}{\mathrm{d}t}\zeta_i(1,t)\Big|_{t=0}.\end{aligned} \tag{3.9}$$

On the other hand, by (3.1), (3.3), and the first identities in (3.6) and (3.7),

$$\begin{aligned}\Big|\frac{D}{\mathrm{d}t}\zeta_i(1,t)\Big|_{t=0}=\sum_{j=1}^{j=k}\big|A_{ij}(1,0)\big|&\le kC'_K\big(|\widetilde{\alpha}(0)|\big)\int_0^1|u_s|_{(s,0)}|u_t|_{(s,0)}\mathrm{d}s\\&\le C_K\big(|\widetilde{\alpha}(0)|\big)|\widetilde{\alpha}(0)||\widetilde{\alpha}'(0)|.\end{aligned}\tag{3.10}$$

The claim follows from (3.9) and (3.10). □

Remark 3.4. Note that (3.3) is applied above with K replaced by the compact set

$$\exp\big(\{v\in T_xM\colon\ x\in K,\ |v|\le|\widetilde{\alpha}(0)|\}\big).$$

Thus, the constants $C'_K(|\widetilde{\alpha}(0)|)$ and $C_K(|\widetilde{\alpha}(0)|)$ may depend on $|\widetilde{\alpha}(0)|$. If M is compact, then the first constant does not depend on $|\widetilde{\alpha}(0)|$, since (3.3) can then be applied with $K=M$. The second constant is then also independent of K and $|\widetilde{\alpha}(0)|$ if $\exp=\exp^{\nabla}$ for some connection ∇ in TM. So, in this case, the function C_K in (3.8) can be taken to be a constant independent of K.

3.2 Poincare lemmas

Lemma 3.5. *If $\zeta\colon S^1\longrightarrow\mathbb{R}^k$ is a smooth function such that $\int_0^{2\pi}\zeta(\theta)\mathrm{d}\theta=0$,*

$$\int_0^{2\pi}|\zeta(\theta)|^2\mathrm{d}\theta\le\int_0^{2\pi}|\zeta'(\theta)|^2\mathrm{d}\theta.$$

Proof. Write

$$\zeta(\theta)=\sum_{n>-\infty}^{n<\infty}\zeta_ne^{\mathrm{i}n\theta};$$

see [6, Section 6.16]. Since ζ integrates to 0, $\zeta_0=0$. Thus,

$$\int_0^{2\pi}|\zeta(\theta)|^2\mathrm{d}\theta=2\pi\sum_{n>-\infty}^{n<\infty}|\zeta_n|^2\le2\pi\sum_{n>-\infty}^{n<\infty}|n\zeta_n|^2=\int_0^{2\pi}|\zeta'(\theta)|^2\mathrm{d}\theta,$$

as claimed. □

Proposition 3.6. *If (M,g) is a Riemannian manifold and $(E,\langle,\rangle,\nabla)$ is a normed vector bundle with connection over M, for every compact subset $K\subset M$ there exists $C_K\in\mathbb{R}^+$ with the following property. If $\alpha\in C^\infty(S^1;M)$ is such that $\operatorname{Im}\alpha\subset K$ and $\xi,\zeta\in\Gamma(\alpha;E)$, then*

$$\big|\langle\!\langle\nabla_\theta\xi,\zeta\rangle\!\rangle\big|\le\|\nabla_\theta\xi\|_2\|\nabla_\theta\zeta\|_2+C_K\min\big(\|\mathrm{d}\alpha\|_1,\|\mathrm{d}\alpha\|_2^2\big)\|\xi\|_{2,1}\|\zeta\|_2\,,$$

where $\nabla_\theta\equiv\nabla^\alpha_{\partial_\theta}$ is the covariant derivative with respect to the oriented unit field on S^1 and all the norms are computed with respect to the standard metric on S^1.

Proof. Identify $E_{\alpha(0)}$ with $\mathbb{R}^k$ (or $\mathbb{C}^k$), preserving the metric. Denote by $so(E_{\alpha(0)})\approx so_k$ (or $u(E_{\alpha(0)})\approx u_k$) the Lie algebra of the Lie group $\mathrm{SO}(E_{\alpha(0)})\approx\mathrm{SO}_k$ (or of $\mathrm{U}(E_{\alpha(0)})\approx\mathrm{U}_k$). For each $\chi\in so(E_{\alpha(0)})$ (or $\chi\in u(E_{\alpha(0)})$), let $\mathrm{e}^\chi\in\mathrm{SO}(E_{\alpha(0)})$ (or $\mathrm{e}^\chi\in\mathrm{U}(E_{\alpha(0)})$) be the exponential of χ. Let

$$\Pi_\theta\colon E_{\alpha(0)}\longrightarrow E_{\alpha(\theta)}$$

be the parallel transport along the curve $t \longrightarrow \alpha(t)$ with $t \in [0,\theta]$. By Corollary 3.2, there exists $\chi \in so(E_{\alpha(0)})$ (or $\chi \in u(E_{\alpha(0)})$) such that

$$\Pi_{2\pi} = \mathrm{e}^{\chi} \qquad \text{and} \qquad |\chi| \le C_K \min\left(\|\mathrm{d}\alpha\|_1, \|\mathrm{d}\alpha\|_2^2\right). \tag{3.11}$$

By the first statement in (3.11),

$$\Psi\colon S^1 \times E_{\alpha(0)} \longrightarrow \alpha^* E\,, \qquad (\theta, v) \longrightarrow \mathrm{e}^{-\theta\chi/2\pi}\Pi_\theta(v),$$

is a smooth isometry. Let $\Phi_2 = \pi_2 \circ \Psi^{-1}\colon \alpha^* E \longrightarrow E_{\alpha(0)}$ and

$$\bar{\zeta} = \frac{1}{2\pi}\int_0^{2\pi} \{\Phi_2\zeta\}(\theta)\mathrm{d}\theta \in E_{\alpha(0)}.$$

By Hölder's inequality and Lemma 3.5,

$$\begin{aligned} \big|\langle\!\langle \nabla_\theta \xi, \zeta - \Psi\bar{\zeta}\rangle\!\rangle\big| &\le \|\nabla_\theta\xi\|_2 \|\zeta - \Psi\bar{\zeta}\|_2 \\ &= \|\nabla_\theta\xi\|_2 \|\Phi_2\zeta - \bar{\zeta}\|_2 \le \|\nabla_\theta\xi\|_2 \|\mathrm{d}(\Phi_2\zeta)\|_2. \end{aligned} \tag{3.12}$$

By the product rule,

$$\begin{aligned} \|\mathrm{d}(\Phi_2\zeta)\|_2 &\le \big\|\mathrm{d}(\Pi^{-1}\zeta)\big\|_2 + |\chi/2\pi|\big\|\Pi^{-1}\zeta\big\|_2 = \|\nabla_\theta\zeta\|_2 + |\chi/2\pi|\|\zeta\|_2 \\ &\le \|\nabla_\theta\zeta\|_2 + C_K \min\left(\|\mathrm{d}\alpha\|_1, \|\mathrm{d}\alpha\|_2^2\right)\|\zeta\|_2. \end{aligned} \tag{3.13}$$

On the other hand, by integration by parts, we obtain

$$\langle\!\langle \nabla_\theta\xi, \zeta - \Psi\bar{\zeta}\rangle\!\rangle = \langle\!\langle \nabla_\theta\xi, \zeta\rangle\!\rangle + \langle\!\langle \xi, \nabla_\theta(\Psi\bar{\zeta})\rangle\!\rangle. \tag{3.14}$$

Since $\Psi\bar{\zeta}$ is the parallel transport of $\mathrm{e}^{\theta\chi/2\pi}\bar{\zeta}$,

$$\begin{aligned} \big|\langle\!\langle \xi, \nabla_\theta(\Psi\bar{\zeta})\rangle\!\rangle\big| &\le \|\xi\|_2 \|\nabla_\theta(\Psi\bar{\zeta})\|_2 = \|\xi\|_2 |\chi/2\pi| \big\|\Psi\bar{\zeta}\big\|_2 \\ &\le C_K \min\left(\|\mathrm{d}\alpha\|_1, \|\mathrm{d}\alpha\|_2^2\right)\|\xi\|_2\|\zeta\|_2. \end{aligned} \tag{3.15}$$

The claim follows from equations (3.12)-(3.15). □

Let $B_{R,r} \subset \mathbb{R}^2$ denote the open annulus with radii $r < R$ centered at the origin.

Corollary 3.7 (of Lemma 3.5)**.** *There exists $C \in C^\infty(\mathbb{R};\mathbb{R})$ such that for all $R \in \mathbb{R}^+$*

$$r \in (0,R], \quad \zeta \in C^\infty\big(B_{R,r};\mathbb{R}^k\big), \quad \int_{B_{R,r}} \zeta = 0 \qquad \Longrightarrow \qquad \|\zeta\|_1 \le C(R/r)R^2\|\mathrm{d}\zeta\|_2.$$

Proof. It is sufficient to assume that $k = 1$. Define

$$\xi\colon S^1 \longrightarrow \mathbb{R} \qquad \text{by} \qquad \xi(\theta) = \int_r^R \zeta(\rho,\theta)\rho\mathrm{d}\rho.$$

By Hölder's inequality and Lemma 3.5,

$$\begin{aligned} \left(\int_0^{2\pi}\left|\int_r^R \zeta(\rho,\theta)\rho\mathrm{d}\rho\right|\mathrm{d}\theta\right)^2 &\le 2\pi\int_0^{2\pi}\big|\xi(\theta)\big|^2\mathrm{d}\theta \le 2\pi\int_0^{2\pi}\big|\xi'(\theta)\big|^2\mathrm{d}\theta \\ &\le 2\pi\int_0^{2\pi}\left(\int_r^R \big|\mathrm{d}_{(\rho,\theta)}\zeta\big|\rho^2\mathrm{d}\rho\right)^2\mathrm{d}\theta \\ &\le \frac{\pi R^4}{2}\int_0^{2\pi}\int_r^R \big|\mathrm{d}_{(\rho,\theta)}\zeta\big|^2\rho\mathrm{d}\rho\mathrm{d}\theta = \frac{\pi R^4}{2}\|\mathrm{d}\zeta\|_2^2\,. \end{aligned} \tag{3.16}$$

If the function $\rho\longrightarrow\zeta(\rho,\theta)$ does not change sign on (r,R), then

$$\int_r^R \big|\zeta(\rho,\theta)\big|\rho\mathrm{d}\rho=\bigg|\int_r^R \zeta(\rho,\theta)\rho\mathrm{d}\rho\bigg|.$$

On the other hand, if this function vanishes somewhere on (r,R), then

$$\big|\zeta(\rho,\theta)\big|\le\int_r^R\big|\mathrm{d}_{(t,\theta)}\zeta\big|\mathrm{d}t\ \ \forall\,\rho\quad\Longrightarrow\quad\int_r^R\big|\zeta(\rho,\theta)\big|\rho\mathrm{d}\rho\le\frac{R^2}{2}\int_r^R\big|\mathrm{d}_{(t,\theta)}\zeta\big|\mathrm{d}t\,.$$

Combining these two cases and using (3.16) and Hölder's inequality, we obtain

$$\begin{aligned}\int_0^{2\pi}\!\!\int_r^R\big|\zeta(\rho,\theta)\big|\rho\mathrm{d}\rho\mathrm{d}\theta&\le\int_0^{2\pi}\bigg|\int_r^R\zeta(\rho,\theta)\rho\mathrm{d}\rho\bigg|\mathrm{d}\theta+\frac{R^2}{2}\int_0^{2\pi}\!\!\int_r^R\big|\mathrm{d}_{(\rho,\theta)}\zeta\big|\mathrm{d}\rho\mathrm{d}\theta\\&\le\frac{\sqrt{\pi}R^2}{\sqrt{2}}\|\mathrm{d}\zeta\|_2+\frac{R^2}{2}\|\mathrm{d}\zeta\|_2\bigg(\int_0^{2\pi}\!\!\int_r^R\rho^{-1}\mathrm{d}\rho\mathrm{d}\theta\bigg)^{1/2}\\&=\sqrt{\frac{\pi}{2}}\Big(1+\sqrt{\ln(R/r)}\Big)R^2\|\mathrm{d}\zeta\|_2\,,\end{aligned}\tag{3.17}$$

as claimed. □

Remark 3.8. By Corollary 4.7 below, C can in fact be chosen to be a constant function. Corollary 3.7 suffices for gluing J-holomorphic maps in symplectic topology, but Corollary 4.7 leads to a sharper version of Proposition 4.14; see Remark 4.13.

3.3 Exponential-like maps and differentiation

Let $(M,g,\exp,\nabla)$ be a smooth Riemannian manifold with an exponential-like map exp and connection ∇ in TM, which is g-compatible, but not necessarily torsion-free. Let

$$T_\nabla\big(\xi(x),\zeta(x)\big)\equiv\big(\nabla_\xi\zeta-\nabla_\zeta\xi-[\xi,\zeta]\big)\big|_x\qquad\forall\,x\!\in\!M,\ \xi,\zeta\!\in\!\Gamma(M;TM),$$

be the torsion tensor of ∇. If $\alpha\colon(-\epsilon,\epsilon)\longrightarrow M$ is a smooth curve and $\xi\in\Gamma(\alpha;TM)$, put

$$\Phi_{\alpha(0)}\Big(\alpha'(0);\xi(0),\frac{D}{\mathrm{d}s}\xi\Big|_{s=0}\Big)=\Pi_{\xi(0)}^{-1}\bigg(\frac{\mathrm{d}}{\mathrm{d}s}\exp\big(\xi(s)\big)\Big|_{s=0}\bigg)=\Pi_{\xi(0)}^{-1}\big(\{\mathrm{d}_{\xi(0)}\exp\}(\xi'(0))\big),$$

where $\xi'(0)\in T_{\xi(0)}(TM)$ is the tangent vector to the curve $\xi\colon(-\epsilon,\epsilon)\longrightarrow TM$ at $s\!=\!0$.

Lemma 3.9. *If $(M,g,\exp,\nabla)$ is a smooth Riemannian manifold with an exponential-like map and a g-compatible connection, there exists $C\in C^\infty(TM;\mathbb{R})$ such that*

$$\Big|\Phi_x(v;w_0,w_1)-\big(v+w_1-T_\nabla(v,w_0)\big)\Big|\le C(w_0)\big(|v||w_0|^2+|w_0||w_1|\big)$$

for all $x\in M$ and $v,w_0,w_1\in T_xM$.

Proof. Let $\alpha\colon(-\epsilon,\epsilon)\longrightarrow M$ be a smooth curve and $\xi\in\Gamma(\alpha;TM)$ such that

$$\alpha(0)=x,\quad\alpha'(0)=v,\quad\xi(0)=w_0,\quad\frac{D}{\mathrm{d}s}\xi(s)\Big|_{s=0}=w_1.$$

Put

$$F_{v,w_0,w_1}(t)=\frac{\mathrm{d}}{\mathrm{d}s}\exp\big(t\xi(s)\big)\Big|_{s=0}=\{\mathrm{d}_{tw_0}\exp\}\big(\mathrm{d}_{w_0}m_t(\xi'(0))\big),$$
$$H_{v,w_0,w_1}(t)=\Pi_{tw_0}\big(v+tw_1-tT_\nabla(v,w_0)\big),$$

where $m_t\colon TM\longrightarrow TM$ is the scalar multiplication by t. Then,

$$F_{v,w_0,w_1}(0)=\frac{\mathrm{d}}{\mathrm{d}s}\alpha(s)\Big|_{s=0}=v=H_{v,w_0,w_1}(0),$$
$$\frac{D}{\mathrm{d}t}F_{v,w_0,w_1}(t)\Big|_{t=0}=\frac{D}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\exp\big(t\xi(s)\big)\Big|_{t=0}\Big|_{s=0}-T_\nabla\big(v,w_0\big)=w_1-T_\nabla(v,w_0)=\frac{D}{\mathrm{d}t}H_{v,w_0,w_1}(t)\Big|_{t=0};$$

see Corollary 3.3. Since

$$F_{\cdot,w_0,\cdot}(t)-H_{\cdot,w_0,\cdot}(t)\in\mathrm{Hom}(T_xM\oplus T_xM,T_{\exp(tw_0)}M),$$

combining the last two equations, we obtain

$$\big|F_{v,w_0,w_1}(t)-H_{v,w_0,w_1}(t)\big|\le C(w_0,t)t^2\big(|v|+|w_1|\big)\quad\forall\ v,w_0,w_1\in T_xM,\ x\in M,\ t\in\mathbb{R},$$

where C is a smooth function on $TM\times\mathbb{R}$. Since

$$F_{v,w_0,w_1}(t)-H_{v,w_0,w_1}(t)=F_{v,tw_0,tw_1}(1)-H_{v,tw_0,tw_1}(1),$$

we conclude that there exists $C\in C^\infty(TM)$ such that

$$\big|F_{v,w_0,w_1}(1)-H_{v,w_0,w_1}(1)\big|\le C(w_0)\big(|w_0|^2|v|+|w_0||w_1|\big)\quad\forall\ v,w_0,w_1\in T_xM,\ x\in M,\tag{3.18}$$

as claimed. $\square$

For any $v,w_0,w_1\in T_xM$, let $\widetilde{\Phi}_x(v;w_0,w_1)=\Phi_x(v;w_0,w_1)-\big(v+w_1-T_\nabla(v,w_0)\big)$.

Corollary 3.10. *If $(M,g,\exp,\nabla)$ is a smooth Riemannian manifold with an exponential-like map and a g-compatible connection, there exists $C\in C^\infty(TM\times_M TM;\mathbb{R})$ such that*

$$\Big|\widetilde{\Phi}_x(v;w_0,w_1)-\widetilde{\Phi}_x(v;w_0',w_1')\Big|$$
$$\le C(w_0,w_0')\Big(\big((|w_0|+|w_0'|)|v|+|w_1|+|w_1'|\big)|w_0-w_0'|+\big(|w_0|+|w_0'|\big)|w_1-w_1'|\Big)$$

for all $x\in M$ and $v,w_0,w_1,w_0',w_1'\in T_xM$.

Proof. By the proof of Lemma 3.9,

$$\widetilde{\Phi}(v;w_0,w_1)=\widetilde{\Phi}_1(w_0;v)+\widetilde{\Phi}_2(w_0;w_1)$$

for some smooth bundle sections $\widetilde{\Phi}_1,\widetilde{\Phi}_2\colon TM\longrightarrow\pi_{TM}^*\mathrm{Hom}(TM,TM)$ such that

$$\big|\widetilde{\Phi}_1(w_0;\cdot)\big|\le C_1(w_0)|w_0|^2,\qquad\big|\widetilde{\Phi}_2(w_0;\cdot)\big|\le C_2(w_0)|w_0|\qquad\forall\ w_0\in TM.$$

Thus,

$$\begin{aligned}\big|\widetilde{\Phi}_1(w_0;\cdot)-\widetilde{\Phi}_1(w_0';\cdot)\big|&\le C_1'(w_0,w_0')\big(|w_0|+|w_0'|\big)|w_0-w_0'|\\ \big|\widetilde{\Phi}_2(w_0;\cdot)-\widetilde{\Phi}_2(w_0';\cdot)\big|&\le C_2'(w_0,w_0')|w_0-w_0'|\end{aligned}\qquad\forall\ w_0,w_0'\in T_xM.$$

From the linearity of $\widetilde{\Phi}_1(w_0;\cdot)$ and $\widetilde{\Phi}_2(w_0;\cdot)$ in the second input, we conclude that

$$\big|\widetilde{\Phi}_1(w_0;v)-\widetilde{\Phi}_1(w_0';v)\big|\le C_1'(w_0,w_0')\big(|w_0|+|w_0'|\big)|w_0-w_0'||v|,$$
$$\Big|\widetilde{\Phi}_2(w_0;w_1)-\widetilde{\Phi}_2(w_0;w_1')\Big|\le C_2'(w_0,w_0')|w_0-w_0'||w_1|+C_2(w_0')|w_0'||w_1-w_1'|.$$

This establishes the claim. $\square$

3.4 Expansion of the $\bar{\partial}$-operator

Let (M,J) and $(\Sigma,\mathfrak{j})$ be almost-complex manifolds. If $u\colon\Sigma\longrightarrow M$ is a smooth map, let

$$\Gamma(u)=\Gamma(\Sigma;u^*TM),\qquad \Gamma^{0,1}_{J,\mathfrak{j}}(u)=\Gamma\big(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}u^*TM\big),$$
$$\bar\partial_{J,\mathfrak{j}}u=\frac{1}{2}\big(\mathrm{d}u+J\circ\mathrm{d}u\circ\mathfrak{j}\big)\in\Gamma^{0,1}_{J,\mathfrak{j}}(u),$$

as in (2.5). If ∇ is a connection in TM, define

$$D^{\nabla}_{J,\mathfrak{j};u}\colon\Gamma(u)\longrightarrow\Gamma^{0,1}_{J,\mathfrak{j}}(u)\qquad\text{by}\qquad D^{\nabla}_{J,\mathfrak{j};u}\xi=\frac{1}{2}\big(\nabla^u\xi+J\nabla^u_{\mathfrak{j}}\xi\big)-\frac{1}{2}\big(T_\nabla(\mathrm{d}u,\xi)+JT_\nabla(\mathrm{d}u\circ\mathfrak{j},\xi)\big).$$

If in addition $\exp\colon TM\longrightarrow M$ is an exponential-like map and $\nabla J=0$, define

$$\exp_u\colon\Gamma(u)\longrightarrow C^\infty(\Sigma;M),\quad \bar\partial_u,N^{\nabla}_{\exp}\colon\Gamma(u)\longrightarrow\Gamma^{0,1}_{J,\mathfrak{j}}(u)\qquad\text{by}$$
$$\big\{\exp_u(\xi)\big\}(z)=\exp\big(\xi(z)\big)\ \ \forall\,z\in\Sigma,\quad \big\{\bar\partial_u\xi\big\}_z(v)=\Pi^{-1}_{\xi(z)}\big(\big\{\bar\partial_{J,\mathfrak{j}}(\exp_u(\xi))\big\}_z(v)\big)\ \ \forall\,z\in\Sigma,\ v\in T_z\Sigma,$$
$$\bar\partial_u\xi=\bar\partial_{J,\mathfrak{j}}u+D^{\nabla}_{J,\mathfrak{j};u}\xi+N^{\nabla}_{\exp}(\xi).$$

Lemma 3.11. *If $(M,J,g,\exp,\nabla)$ is an almost-complex Riemannian manifold with an exponential-like map and a g-compatible connection in (TM,J), there exists $C\in C^\infty(TM\times_M TM;\mathbb{R})$ with the following property. If $(\Sigma,\mathfrak{j})$ is an almost complex manifold, $u:\Sigma\longrightarrow M$ is a smooth map, and $\xi,\xi'\in\Gamma(u)$, then*

$$\begin{aligned}\Big|\big\{N^{\nabla}_{\exp}(\xi)\big\}_z(v)-\big\{N^{\nabla}_{\exp}(\xi')\big\}_z(v)\Big|\le C\big(\xi(z),\xi'(z)\big)\Big(\big(|\xi(z)|+|\xi'(z)|\big)\big(|\nabla_v(\xi-\xi')|+|\nabla_{\mathfrak{j}v}(\xi-\xi')|\big)\\ +\big((|\mathrm{d}_zu(v)|+|\mathrm{d}_zu(\mathfrak{j}v)|)(|\xi(z)|+|\xi'(z)|)+(|\nabla_v\xi|+|\nabla_{\mathfrak{j}v}\xi|+|\nabla_v\xi'|+|\nabla_{\mathfrak{j}v}\xi|)\big)\big|\xi(z)-\xi'(z)\big|\Big)\end{aligned}$$

for all $z\in\Sigma$, $v\in T_z\Sigma$. Furthermore, $N^{\nabla}_{\exp}(0)=0$.

Proof. Since the connection ∇ commutes with J, so does the parallel transport Π. Thus, with notation as in Section 3.3,

$$\big\{N^{\nabla}_{\exp}(\xi)\big\}_z(v)=\frac{1}{2}\Big(\widetilde{\Phi}\big(\mathrm{d}_zu(v);\xi(z),\nabla_v\xi\big)+J\big(u(z)\big)\widetilde{\Phi}\big(\mathrm{d}_zu(\mathfrak{j}v);\xi(z),\nabla_{\mathfrak{j}v}\xi\big)\Big).$$

The claim now follows from Corollary 3.10. $\square$

Definition 3.12. *Let M be a smooth manifold and $(E,\langle,\rangle,\nabla)$ a normed vector bundle with connection over M. If $C_0\in\mathbb{R}^+$, $(\Sigma,\mathfrak{j})$ is an almost complex manifold, and $u:\Sigma\longrightarrow M$ is a smooth map, norms $\|\cdot\|_{p,1}$ and $\|\cdot\|_p$ on $\Gamma(u;E)$ and $\Gamma^1(u;E)$, respectively, are C_0-*admissible* if for all $\xi\in\Gamma(u;E)$, $\eta\in\Gamma^1(u;E)$, and every continuous function $f:\Sigma\longrightarrow\mathbb{R}$,*

$$\|f\eta\|_p\le\|f\|_{C^0}\|\eta\|_p,\quad \|\eta\circ\mathfrak{j}\|_p=\|\eta\|_p,\quad \|\nabla^u\xi\|_p\le\|\xi\|_{p,1},\quad \|\xi\|_{C^0}\le C_0\|\xi\|_{p,1}.$$

Proposition 3.13. *If $(M,J,g,\exp,\nabla)$ is an almost-complex Riemannian manifold with an exponential-like map and a g-compatible connection in (TM,J), for every compact subset $K\subset M$ there exists $C_K\in C^\infty(\mathbb{R};\mathbb{R})$ with the following property. If $(\Sigma,\mathfrak{j})$ is an almost complex manifold, $u:\Sigma\longrightarrow K$ is a smooth map, and $\|\cdot\|_{p,1}$ and $\|\cdot\|_p$ are C_0-admissible norms on $\Gamma(u;TM)$ and $\Gamma^1(u;TM)$, respectively, then*

$$\big\|N^{\nabla}_{\exp}(\xi)-N^{\nabla}_{\exp}(\xi')\big\|_p\le C_K\big(C_0+\|\mathrm{d}u\|_p+\|\xi\|_{p,1}+\|\xi'\|_{p,1}\big)\big(\|\xi\|_{p,1}+\|\xi'\|_{p,1}\big)\|\xi-\xi'\|_{p,1}$$

for all $\xi,\xi'\in\Gamma(u)$. Furthermore, $N^{\nabla}_{\exp}(0)=0$. If the g-ball $B_{g;\delta}(u(z))$ of radius δ around $f(z)$ for some $z\in\Sigma$ is isomorphic to an open subset of $\mathbb{C}^n$ and $|\xi(z)|<\delta$, then $\{N^{\nabla}_{\exp}\xi\}_z=0$.

Proof. The first two statements follow from Lemma 3.11 and Definition 3.12. The last claim is clear from the definition of $N_{\exp}^{\nabla}$. $\Box$

Remark 3.14. As the notation suggests, one possibility for the norms $\|\cdot\|_{p,1}$ and $\|\cdot\|_p$ is the usual Sobolev L^p_1 and L^p-norms with respect to some Riemannian metric on Σ, where $p>\dim_{\mathbb{R}}\Sigma$. Another natural possibility in the $\dim_{\mathbb{R}}\Sigma=2$ case is the modified Sobolev norms introduced in [3, Section 3]; these are particularly suited for gluing pseudo-holomorphic curves. By Proposition 4.10 below, in the $\dim_{\mathbb{R}}\Sigma=2$ case the constant C_0 itself is a function of $\|\mathrm{d}u\|_p$ only for either of these two choices of norms.

Remark 3.15. By Proposition 3.13, the operator $D^{\nabla}_{J,\mathfrak{j};u}$ defined above is a linearization of the $\bar\partial$-operator on the space of smooth maps to M at u. If ∇' is any connection in TM, the connection

$$\nabla\colon \Gamma(M;TM)\longrightarrow\Gamma(M;T^*M\otimes_{\mathbb{R}}TM),\quad \nabla_v\xi=\frac{1}{2}\Big(\nabla'_v\xi-J\nabla'_v(J\xi)\Big)\quad\forall\,v\in TM,\,\xi\in\Gamma(M;TM),$$

is J-compatible. If in addition ∇' and J are compatible with a Riemannian metric g on M, then so is ∇. If ∇' is also the Levi-Civita connection of the metric g (i.e. $T_{\nabla'}=0$),

$$T_{\nabla}(v,w)=\frac{1}{2}\big(J(\nabla'_wJ)v-J(\nabla'_vJ)w\big)\qquad\forall\,v,w\in T_xM,\,x\in M.$$

If the 2-form $\omega(\cdot,\cdot)\equiv g(J\cdot,\cdot)$ is closed as well, then

$$\nabla'_{Jv}J=-J\nabla'_vJ\qquad\forall\,v\in TM$$

by [4, (C.7.5)] and thus

$$T_{\nabla}(v,w)=-\frac{1}{4}\big(J(\nabla'_vJ)w-J(\nabla'_wJ)v-(\nabla'_{Jv}J)w+(\nabla'_{Jw}J)v\big)=-A_J(v,w)\quad\forall\,v,w\in T_xM,\,x\in M,$$

where A_J is the Nijenhuis tensor of J as in (2.27). The operator $D^{\nabla}_{J,\mathfrak{j};u}$ then becomes

$$D^{\nabla}_{J,\mathfrak{j};u}\colon\Gamma(u)\longrightarrow\Gamma^{0,1}_{J,\mathfrak{j}}(u),\qquad D^{\nabla}_{J,\mathfrak{j};u}\xi=\bar\partial_{\nabla^u}\xi+A_J(\partial_{J,\mathfrak{j}}u,\xi),\tag{3.19}$$

where

$$\begin{aligned}\bar\partial_{\nabla^u}\xi&=\frac{1}{2}\big(\nabla^u\xi+J\nabla^u_{\mathfrak{j}}\xi\big)\in\Gamma^{0,1}_{J,\mathfrak{j}}(u),\\ \partial_{J,\mathfrak{j}}u&=\frac{1}{2}\big(\mathrm{d}u-J\circ\mathrm{d}u\circ\mathfrak{j}\big)\in\Gamma\big(\Sigma;T^*\Sigma^{1,0}\otimes_{\mathbb{C}}u^*TM\big).\end{aligned}$$

This agrees with [4, (3.1.5)], since the Nijenhuis tensor of J is defined to be $-4A_J$ in [4, p18].

4 Sobolev and elliptic inequalities

This appendix refines, in the $n=2$ case, the proofs of Sobolev Embedding Theorems given in [5] to obtain a C^0-estimate in Proposition 4.10 and elliptic estimates for the $\bar\partial$-operator in Propositions 4.14 and 4.16. If $R,r\in\mathbb{R}$, let

$$B_R=\big\{x\in\mathbb{R}^2\colon\,|x|<R\big\},\qquad B_{R,r}=B_R-\bar B_r\,,\qquad \widetilde{B}_{R,r}=B_R-B_r\,.$$

4.1 Eucledian case

If ξ is an $\mathbb{R}^k$-valued function defined on a subset B of $\mathbb{R}^2$, let $\mathrm{supp}_{\mathbb{R}^2}(\xi)$ be the closure of $\mathrm{supp}(\xi)\subset B$ in $\mathbb{R}^2$. If U is an open subset of $\mathbb{R}^2$, $\xi\in C^\infty(U;\mathbb{R}^k)$, and $p\geq 1$, let

$$\|\xi\|_p \equiv \bigg(\int_U |\xi|^p\bigg)^{1/p}, \qquad \|\xi\|_{p,1}\equiv \|\xi\|_p+\|\mathrm{d}\xi\|_p\,,$$

be the usual Sobolev norms of ξ.

Lemma 4.1. *For every bounded convex domain $\mathcal{D}\subset\mathbb{R}^2$, $\xi\in C^\infty(\mathcal{D};\mathbb{R}^k)$, and $x\in\mathcal{D}$,*

$$\big|\xi_{\mathcal{D}}-\xi(x)\big| \le \frac{2r_0^2}{|\mathcal{D}|}\int_{\mathcal{D}}|\mathrm{d}_y\xi||y-x|^{-1}\mathrm{d}y,$$

where $2r_0$ is the diameter of $\mathcal{D}$, $|\mathcal{D}|$ is the area of $\mathcal{D}$, and

$$\xi_{\mathcal{D}}=\frac{1}{|\mathcal{D}|}\Big(\int_{\mathcal{D}}\xi(y)\mathrm{d}y\Big)$$

is the average value of ξ on $\mathcal{D}$.

Proof. For any $y\in\mathcal{D}$,

$$\xi(y)-\xi(x)=\int_0^1\frac{\mathrm{d}}{\mathrm{d}t}\xi\big(x+t(y-x)\big)\mathrm{d}t=\int_0^1\mathrm{d}_{x+t(y-x)}\xi(y-x)\mathrm{d}t.$$

Putting $g(z)=|\mathrm{d}_z\xi|$ if $z\in\mathcal{D}$ and $g(z)=0$ otherwise, we obtain

$$\big|\xi_{\mathcal{D}}-\xi(x)\big|\le\frac{1}{|\mathcal{D}|}\int_{y\in\mathcal{D}}|\xi(y)-\xi(x)|\mathrm{d}y\le\frac{1}{|\mathcal{D}|}\int_{y\in\mathcal{D}}\int_0^\infty g\big(x+t(y-x)\big)|y-x|\mathrm{d}t\mathrm{d}y.$$

Rewriting the last integral in polar coordinates (r,θ) centered at x, we obtain

$$\begin{aligned}\big|\xi_{\mathcal{D}}-\xi(x)\big| &\le \frac{1}{|\mathcal{D}|}\int_0^{2\pi}\int_0^{2r_0}\int_0^\infty g(tr,\theta)r^2\mathrm{d}t\mathrm{d}r\mathrm{d}\theta\\ &=\frac{1}{|\mathcal{D}|}\int_0^{2\pi}\int_0^{2r_0}\int_0^\infty g(t,\theta)r\mathrm{d}t\mathrm{d}r\mathrm{d}\theta=\frac{2r_0^2}{|\mathcal{D}|}\int_0^{2\pi}\int_0^\infty g(t,\theta)\mathrm{d}t\mathrm{d}\theta\\ &=\frac{2r_0^2}{|\mathcal{D}|}\int_{\mathcal{D}}|\mathrm{d}_y\xi||y-x|^{-1}\mathrm{d}y.\end{aligned}$$

This establishes the claim. □

Corollary 4.2. *For every $p>2$, there exists $C_p>0$ such that*

$$r\in\big[0,R/2\big],\quad \xi\in C^\infty(B_{R,r};\mathbb{R}^k)\qquad\Longrightarrow\qquad \big|\xi(x)-\xi(y)\big|\le C_pR^{\frac{p-2}{p}}\|\mathrm{d}\xi\|_p\quad\forall\,x,y\in B_{R,r}\,.$$

Proof. For any $x\in B_{R,r}$, put

$$\mathcal{D}_x=\big\{y\in B_{R,r}\colon\langle x,|x|y-rx\rangle>0\big\}.$$

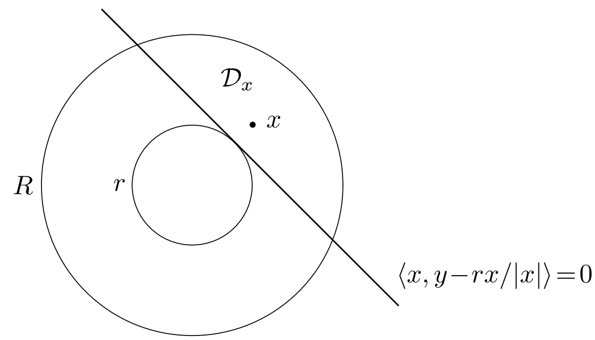

Figure 2: A convex region $\mathcal{D}_x$ of the annulus $\mathcal{D}_{R,r}$ containing x

If $x\neq 0$, $\mathcal{D}_x$ is the part of the annulus on the same side of the line $\langle x, y-rx/|x|\rangle=0$ as x; see Figure 2. In particular,

$$\operatorname{diam}(\mathcal{D}_x)\le 2R\,,\qquad |\mathcal{D}_x|\ge\Big(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\Big)R^2.$$

Thus, by Lemma 4.1 and Hölder's inequality,

$$\begin{aligned}\big|\xi(x)-\xi_{\mathcal{D}_x}\big|&\le 12\int_{y\in\mathcal{D}_x}|\mathrm{d}_y\xi||y-x|^{-1}\mathrm{d}y\\&\le 12\bigg(\int_{y\in B_{2R}(x)}|y-x|^{-\frac{p}{p-1}}\bigg)^{\frac{p-1}{p}}\|\mathrm{d}\xi\|_p\le C_pR^{\frac{p-2}{p}}\|\mathrm{d}\xi\|_p,\end{aligned}\tag{4.1}$$

since $\frac{p}{p-1}<2$. Let

$$x_\pm=\big(\pm(R-r)/2,0\big),\qquad y_\pm=\big(0,\pm(R-r)/2\big).$$

Since each of the convex regions $\mathcal{D}_{x_\pm}$ intersects $\mathcal{D}_{y_+}$ and $\mathcal{D}_{y_-}$ and $\mathcal{D}_x$ intersects at least one (in fact precisely two if $r\neq0$) of these four convex regions for every $x\in B_{R,r}$,

$$\big|\xi(x)-\xi(y)\big|\le 8C_pR^{\frac{p-2}{p}}\|\mathrm{d}\xi\|_p\quad\forall\,x,y\in B_{R,r}$$

by (4.1) and triangle inequality. □

Corollary 4.3. *For every $p>2$, there exists $C_p\in C^\infty(\mathbb{R}^+;\mathbb{R})$ such that*

$$r\in\big[0,R/2\big],\quad \xi\in C^\infty(B_{R,r};\mathbb{R}^k)\quad\Longrightarrow\quad \|\xi\|_{C^0}\le C_p(R)\|\xi\|_{p,1}.$$

Proof. By Corollary 4.2 and Hölder's inequality, for every $x\in B_{R,r}$

$$\begin{aligned}|\xi(x)|&\le\big|\xi_{B_{R,r}}\big|+C_pR^{\frac{p-2}{p}}\|\mathrm{d}\xi\|_p\le\frac{1}{|B_{R,r}|}\|\xi\|_1+C_pR^{\frac{p-2}{p}}\|\mathrm{d}\xi\|_p\\&\le|B_{R,r}|^{-\frac{1}{p}}\|\xi\|_p+C_pR^{\frac{p-2}{p}}\|\mathrm{d}\xi\|_p\le(1+C_p)R^{-\frac{2}{p}}\big(\|\xi\|_p+R\|\mathrm{d}\xi\|_p\big).\end{aligned}\tag{4.2}$$

This implies the claim. □

Lemma 4.4. *For all $R>0$ and $r\in[0,R)$,*

$$\zeta\in C^\infty(B_{R,r};\mathbb{R}^k),\quad \mathrm{supp}_{\mathbb{R}^2}(\zeta)\subset\widetilde{B}_{R,r}\quad\Longrightarrow\quad \|\zeta\|_2\le\|\mathrm{d}\zeta\|_1.$$

Proof. Such a function ζ can be viewed as a function on the complement of the ball B_r in $\mathbb{R}^2$. Since ζ vanishes at infinity, for any $(x,y)\in B_{R,r}$

$$\zeta(x,y)=\begin{cases}\int_{-\infty}^x\zeta_s(s,y)\mathrm{d}s, & \text{if } x\le 0;\\ -\int_x^\infty\zeta_s(s,y)\mathrm{d}s, & \text{if } x\ge 0;\end{cases}\qquad \zeta(x,y)=\begin{cases}\int_{-\infty}^y\zeta_t(x,t)\mathrm{d}t, & \text{if } y\le 0;\\ -\int_y^\infty\zeta_t(x,t)\mathrm{d}t, & \text{if } y\ge 0.\end{cases}$$

Taking the absolute value in these equations, we obtain

$$\big|\zeta(x,y)\big|\le\int_{-\infty}^\infty\big|\mathrm{d}_{(s,y)}\zeta\big|\mathrm{d}s\quad\text{and}\quad\big|\zeta(x,y)\big|\le\int_{-\infty}^\infty\big|\mathrm{d}_{(x,t)}\zeta\big|\mathrm{d}t,\tag{4.3}$$

where we formally set ζ and $\mathrm{d}\zeta$ to be zero on the smaller disk. Multiplying the two inequalities in (4.3) and integrating with respect to x and y, we conclude

$$\int_{-\infty}^\infty\int_{-\infty}^\infty\big|\zeta(x,y)\big|^2\mathrm{d}x\mathrm{d}y\le\Big(\int_{-\infty}^\infty\int_{-\infty}^\infty\big|\mathrm{d}_{(x,y)}\zeta\big|\mathrm{d}x\mathrm{d}y\Big)^2,$$

as claimed. □

Corollary 4.5. *For all $p,q\ge1$ with $1-2/p\ge-2/q$, there exists $C_{p,q}\in\mathbb{R}^+$ such that*

$$r\in[0,R),\quad \xi\in C^\infty(B_{R,r};\mathbb{R}^k),\quad \mathrm{supp}_{\mathbb{R}^2}(\xi)\subset\widetilde{B}_{R,r}\quad\Longrightarrow\quad\|\xi\|_q\le C_{p,q}R^{1-\frac{2}{p}+\frac{2}{q}}\|\mathrm{d}\xi\|_p.$$

Proof. We can assume that $k=1$. For $\epsilon>0$, let $\zeta_\epsilon=(\xi^2+\epsilon)^{\frac{q}{4}}-\epsilon^{\frac{q}{4}}$. By Lemma 4.4 and Hölder's inequality,

$$\begin{aligned}\|\xi\|_q^q&\le\big\|\zeta_\epsilon+\epsilon^{\frac{q}{4}}\big\|_2^2\le2\|\mathrm{d}\zeta_\epsilon\|_1^2+2\epsilon^{\frac{q}{2}}\pi R^2=2\Big\|\frac{q}{2}(\xi^2+\epsilon)^{\frac{q}{4}-1}\xi\mathrm{d}\xi\Big\|_1^2+2\epsilon^{\frac{q}{2}}\pi R^2\\&\le q^2\big\|(\xi^2+\epsilon)^{\frac{q}{4}-\frac{1}{2}}\mathrm{d}\xi\big\|_1^2+2\epsilon^{\frac{q}{2}}\pi R^2\le q^2\|\mathrm{d}\xi\|_p^2\big\|(\xi^2+\epsilon)^{\frac{q-2}{4}}\big\|_{\frac{p}{p-1}}^2+2\epsilon^{\frac{q}{2}}\pi R^2.\end{aligned}\tag{4.4}$$

Note that

$$1-\frac{2}{p}=-\frac{2}{q}\quad\Longrightarrow\quad\frac{q-2}{4}\frac{p}{p-1}=\frac{q-2}{4}\frac{2q}{q-2}=\frac{q}{2}.$$

Thus, letting ϵ go to zero in (4.4), we obtain

$$\|\xi\|_q^q\le q^2\|\mathrm{d}\xi\|_p^2\|\xi\|_q^{q-2}\quad\Longrightarrow\quad\|\xi\|_q\le q\|\mathrm{d}\xi\|_p.$$

The case $1-\frac{2}{p}>-\frac{2}{q}$ follows by Hölder's inequality. □

Remark 4.6. By Hölder's inequality, the constant $C_{p,q}$ can be taken to be

$$C_{p,q}=\max(2,q)\pi^{\frac{1}{2}\left(1-\frac{2}{p}+\frac{2}{q}\right)}.$$

Corollary 4.7 (of Lemmas 4.1, 4.4)**.** *There exists $C>0$ such that for all $R\in\mathbb{R}^+$*

$$r\in[0,R],\quad\zeta\in C^\infty\big(B_{R,r};\mathbb{R}^k\big),\quad\int_{B_{R,r}}\zeta=0\qquad\Longrightarrow\qquad\|\zeta\|_1\le CR^2\|\mathrm{d}\zeta\|_2.$$

Proof. (1) If $\zeta\in C^\infty(B_{R,r};\mathbb{R}^k)$ integrates to 0 over its domain, then so does the function

$$\widetilde{\zeta}\in C^\infty\big(B_{1,r/R};\mathbb{R}^k\big), \qquad \widetilde{\zeta}(z)=\zeta(Rz).$$

Furthermore, $\|\widetilde{\zeta}\|_1=\|\zeta\|_1/R^2$ and $\|\mathrm{d}\widetilde{\zeta}\|_2=\|\mathrm{d}\zeta\|_2$. Thus, it is sufficient to prove the claim for $R=1$.

(2) If $r=0$, for some open half-disk $\mathcal{D}\subset B_{1,0}$

$$\int_{\mathcal{D}}\zeta=0, \qquad \big\|\zeta|_{\mathcal{D}}\big\|_1\ge\frac{1}{2}\|\zeta\|_1\,. \tag{4.5}$$

By the first condition, Lemma 4.1, and Hölder's inequality

$$\big\|\zeta|_{\mathcal{D}}\big\|_1\le\frac{4}{\pi}\int_{\mathcal{D}}\int_{\mathcal{D}}|\mathrm{d}_y\zeta||y-x|^{-1}\mathrm{d}y\mathrm{d}x\le 16\int_{\mathcal{D}}|\mathrm{d}_y\zeta|\mathrm{d}y\le 8\sqrt{2\pi}\|\mathrm{d}\zeta\|_2\,.$$

Along with the second assumption in (4.5), this implies the claim for $r=0$ with $C=16\sqrt{2\pi}$.

(3) Let $\beta\colon\mathbb{R}\longrightarrow[0,1]$ be a smooth function such that

$$\beta(t)=\begin{cases}1, & \text{if } t\le 1/2;\\ 0, & \text{if } t\ge 1.\end{cases}$$

It remains to prove the claim for all $r>0$ and $R=1$. By (3.17), we can assume that

$$r\le\frac{1}{48\sqrt{3\pi}\|\beta'\|_{C^0}}<\frac{1}{96\sqrt{3\pi}}\,. \tag{4.6}$$

We first consider the case

$$\big\|\zeta|_{B_{2r,r}}\big\|_1\ge\frac{1}{25}\|\zeta\|_1. \tag{4.7}$$

Using polar coordinates, define $\widetilde{\zeta}\in C^\infty(B_{1,r};\mathbb{R}^k)$ by

$$\widetilde{\zeta}(\rho,\theta)=\beta(\rho)\zeta(\rho,\theta).$$

By Hölder's inequality and Lemma 4.4,

$$\big\|\zeta|_{B_{2r,r}}\big\|_1\le\sqrt{3\pi}r\|\widetilde{\zeta}\|_2\le\sqrt{3\pi}r\|\mathrm{d}\widetilde{\zeta}\|_1\le\sqrt{3\pi}r\big(\|\mathrm{d}\zeta\|_1+\|\beta'\|_{C^0}\big\|\zeta|_{B_{1,1/2}}\big\|_1\big).$$

Along with the assumptions (4.6) and (4.7), this implies the bound with

$$C=25\frac{\sqrt{3\pi}r}{1-24\sqrt{3\pi}\|\beta\|_{C^0}r}\le\frac{25}{48}\,.$$

Finally, suppose

$$\big\|\zeta|_{B_{2r,r}}\big\|_1\le\frac{1}{25}\|\zeta\|_1. \tag{4.8}$$

Split the annulus $B_{1,r}$ into 3 wedges of equal area; split each wedge into a large convex outer portion and a small inner portion by drawing the line segment tangent to the circle of radius r and with the end points on the sides of the wedges $2r$ from the center as in Figure 3. By (4.8),

$$A\equiv\big\|\zeta|_{\mathcal{D}_+}\big\|_1\ge\frac{8}{25}\|\zeta\|_1 \tag{4.9}$$

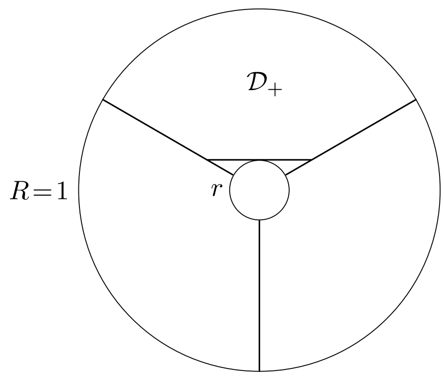

Figure 3: A large convex region $\mathcal{D}_+$ of an annulus $\mathcal{D}$

for the outer piece $\mathcal{D}_+$ of some wedge $\mathcal{D}$. If

$$\bigg|\int_{\mathcal{D}_+}\zeta\bigg| \le \frac{3}{10}A\,,$$

then by Lemma 4.1, (4.6), and Hölder's inequality,

$$\begin{aligned}
A &\le \frac{3}{10}A + \frac{2\left(\frac{\sqrt{3}}{2}\right)^2}{\frac{\pi}{3}\left(1-\left(\frac{1}{96\sqrt{3\pi}}\right)^2\right)}\int_{\mathcal{D}_+}\int_{\mathcal{D}_+}|\mathrm{d}_y\zeta||y-x|^{-1}\mathrm{d}y\mathrm{d}x\\
&\le \frac{3}{10}A + \frac{9}{2\pi}\cdot\frac{7\sqrt{2}}{9}\cdot 2\pi\sqrt{3}\int_{\mathcal{D}}|\mathrm{d}_y\zeta|\mathrm{d}y \le \frac{3}{10}A + 7\sqrt{2\pi}\|\mathrm{d}\zeta\|_2\,.
\end{aligned}$$

Along with the assumption (4.9), this implies the bound with $C\!=\!125\sqrt{2\pi}/4$. If

$$\bigg|\int_{\mathcal{D}_+}\zeta\bigg| \ge \frac{3}{10}A\,,$$

then by (4.8), (4.9), and (3.16),

$$\begin{aligned}
A \le \big\|\xi|_{\mathcal{D}}\big\|_1 &\le \big\|\zeta|_{\mathcal{D}}\big\|_1 - \bigg|\int_{\mathcal{D}}\zeta\bigg| + \int_0^{2\pi}\bigg|\int_r^1\zeta(\rho,\theta)\rho\mathrm{d}\rho\bigg|\mathrm{d}\theta\\
&\le \left(A+\frac{1}{8}A\right) - \left(\frac{3}{10}A-\frac{1}{8}A\right) + \sqrt{\frac{\pi}{2}}\|\mathrm{d}\zeta\|_2 = \frac{19}{20}A + \sqrt{\frac{\pi}{2}}\|\mathrm{d}\zeta\|_2\,.
\end{aligned}$$

Along with the assumption (4.9), this implies the bound with $C = 125\sqrt{2\pi}/4$. Since β can be chosen so that $\|\beta'\|_{C^0}\!<\!3$ (actually arbitrarily close to 2), comparing with (3.17) for $R/r\!=\!144\sqrt{3\pi}$ we conclude that the claim holds with $C\!=\!125\sqrt{2\pi}/4$ for all r. $\square$

4.2 Bundle sections along smooth maps

Let (M,g) be a Riemannian manifold and $(E,\langle,\rangle,\nabla)$ a normed vector bundle with connection over M. If $u\!\in\!C^\infty(\widetilde{B}_{R,r};M)$, $\xi\!\in\!\Gamma(u;E)$, and $p\!\ge\!1$, let

$$\|\xi\|_p \equiv \left(\int_{\widetilde{B}_{R,r}}|\xi|^p\right)^{1/p}, \qquad \|\xi\|_{p,1} \equiv \|\xi\|_p + \|\nabla^u\xi\|_p\,.$$

Lemma 4.8. *If (M,g) is a Riemannian manifold, $(E,\langle,\rangle,\nabla)$ is a normed vector bundle with connection over M, and $p,q\ge 1$ are such that $1-2/p\ge -2/q$, for every compact subset $K\subset M$ there exists $C_{K;p,q}\in\mathbb{R}^+$ with the following property. If $R\in\mathbb{R}^+$, $r\in[0,R)$, $u\in C^\infty(\widetilde{B}_{R,r};M)$ is such that $\operatorname{Im} u\subset K$, and $\xi\in\Gamma_c(u;E)$, then*

$$\|\xi\|_q \le C_{K;p,q}R^{1-\frac{2}{p}+\frac{2}{q}}\big(\|\nabla^u\xi\|_p+\|\xi\otimes \mathrm{d}u\|_p\big).$$

Proof. Let $\exp\colon TM\longrightarrow M$ be an exponential-like map and $\{U_i\colon i\in[N]\}$ a finite open cover of K such that the g-diameter of each set U_i is at most $r^g_{\exp}(K)/2$. Let $\{W_i\colon i\in[N]\}$ be an open cover of K such that $\overline{W}_i\subset U_i$. Choose smooth functions $\eta_i\colon M\longrightarrow[0,1]$ such that $\eta_i=1$ on W_i and $\eta_i=0$ outside of U_i. For each $i\in[N]$, pick $x_i\in W_i$. For each $z\in u^{-1}(U_i)\subset\widetilde{B}_{R,r}$, define $\widetilde{u}_i(z)\in T_{x_i}M$ and $\xi_i(z)\in E_{x_i}$ by

$$\exp_{x_i}\widetilde{u}_i(z)=u(z),\quad |\widetilde{u}_i(z)|<r_{\exp}(x_i);\qquad \Pi_{\widetilde{u}_i(z)}\xi_i(z)=\xi(z).$$

For any $z\in B_{R,r}$, put $\widetilde{\xi}_i(z)=\eta_i(u(z))\xi_i(z)$. Since $\widetilde{\xi}_i\in C^\infty_c(\widetilde{B}_{R,r};E_{x_i})$, by Corollary 4.5 there exists $C_{i;p,q}>0$ such that

$$\big\|\xi|_{u^{-1}(W_i)}\big\|_q=\big\|\widetilde{\xi}_i|_{u^{-1}(W_i)}\big\|_q\le\|\widetilde{\xi}_i\|_q\le C_{i;p,q}R^{1-\frac{2}{p}+\frac{2}{q}}\|\mathrm{d}\widetilde{\xi}_i\|_p\,. \tag{4.10}$$

Since $\mathrm{d}\widetilde{\xi}_i=(\mathrm{d}\eta_i\circ\mathrm{d}u)\xi_i+(\eta\circ u)\mathrm{d}\xi_i$ on $u^{-1}(U_i)$ and vanishes outside of $u^{-1}(U_i)$,

$$\|\mathrm{d}\widetilde{\xi}_i\|_p\le\big\|\mathrm{d}\xi_i|_{u^{-1}(U_i)}\big\|_p+C_i\|\xi_i\otimes\mathrm{d}u\|_p. \tag{4.11}$$

On the other hand, by Corollary 3.3, if $u(z)\in U_i$

$$\Big|\nabla^u\xi|_z-\Pi_{\widetilde{u}_i(z)}\circ\mathrm{d}_z\xi_i\Big|\le C_K|\mathrm{d}_zu||\xi(z)|. \tag{4.12}$$

Combining equations (4.10)-(4.12), we obtain

$$\big\|\xi|_{u^{-1}(W_i)}\big\|_q\le\widetilde{C}_{i;p,q}R^{1-\frac{2}{p}+\frac{2}{q}}\big(\|\xi\|_{p,1}+\|\xi\otimes\mathrm{d}u\|_p\big).$$

The claim follows by summing the last inequality over all i. □

Lemma 4.9. *If (M,g) is a Riemannian manifold, $(E,\langle,\rangle,\nabla)$ is a normed vector bundle with connection over M, and $p>2$, for every compact subset $K\subset M$ there exists $C_{K;p}\in C^\infty(\mathbb{R}^+;\mathbb{R})$ with the following property. If $R\in\mathbb{R}^+$, $r\in[0,R/2]$, $u\in C^\infty(B_{R,r};M)$ is such that $\operatorname{Im} u\subset K$, and $\xi\in\Gamma(u;E)$, then*

$$\|\xi\|_{C^0}\le C_{K;p}(R)\big(\|\xi\|_{p,1}+\|\xi\otimes\mathrm{d}u\|_p\big).$$

Proof. We continue with the setup in the proof of Lemma 4.8. By Corollary 4.3,

$$\big\|\xi|_{u^{-1}(W_i)}\big\|_{C^0}\le\|\widetilde{\xi}_i\|_{C^0}\le C_{i;p}(R)\|\widetilde{\xi}_i\|_{p,1}\le C_{i;p}(R)\big(\big\|\xi|_{u^{-1}(U_i)}\big\|_p+\|\mathrm{d}\widetilde{\xi}_i\|_p\big).$$

As above, we obtain

$$\|\mathrm{d}\widetilde{\xi}_i\|_p\le C_i\big(\|\nabla^u\xi\|_p+\|\xi\otimes\mathrm{d}u\|_p\big),$$

and the claim follows. □

Proposition 4.10. *If (M,g) is a Riemannian manifold, $(E,\langle,\rangle,\nabla)$ is a normed vector bundle with connection over M, and $p>2$, for every compact subset $K\subset M$ there exists $C_{K;p}\in C^\infty(\mathbb{R}^+\times\mathbb{R};\mathbb{R})$ with the following property. If $R\in\mathbb{R}^+$, $r\in[0,R/2]$, $u\in C^\infty(B_{R,r};M)$ is such that $\operatorname{Im}u\subset K$, and $\xi\in\Gamma_c(u;E)$, then*

$$\|\xi\|_{C^0}\le C_{K;p}\big(R,\|\mathrm{d}u\|_p\big)\|\xi\|_{p,1}.$$

The same statement holds if $B_{R,r}$ is replaced by a fixed compact Riemann surface (Σ,g_Σ).

Proof. By Lemma 4.9 applied with $\widetilde{p}=(p+2)/2$ and Hölder's inequality,

$$\|\xi\|_{C^0}\le C_{K;\widetilde{p}}(R)\big(\|\xi\|_{\widetilde{p},1}+\|\xi\otimes\mathrm{d}u\|_{\widetilde{p}}\big)\le \widetilde{C}_{K;\widetilde{p}}(R)\big(\|\xi\|_{p,1}+\|\mathrm{d}u\|_p\|\xi\|_{q_1}\big), \tag{4.13}$$

where $q_1=p(p+2)/(p-2)$. If $q_1\le p$, then the proof is complete. Otherwise, apply Lemma 4.8 with $p_1=2q_1/(q_1+2)$ and Hölder's inequality:

$$\|\xi\|_{q_1}\le C_{K;p_1,q_1}(R)\big(\|\xi\|_{p_1,1}+\|\xi\otimes\mathrm{d}u\|_{p_1}\big)\le C_{K;1}(R)\big(\|\xi\|_{p,1}+\|\mathrm{d}u\|_p\|\xi\|_{q_2}\big), \tag{4.14}$$

where $q_2=pp_1/(p-p_1)$. If $q_2\le p$, then the claim follows from equations (4.13) and (4.14). Otherwise, we can continue and construct sequences $\{p_i\},\{q_i\},\{C_{K;i}\}$ such that

$$p_i=\frac{2q_i}{q_i+2},\quad q_{i+1}=\frac{pp_i}{p-p_i}; \tag{4.15}$$

$$\|\xi\|_{q_i}\le C_{K;i}(R)\big(\|\xi\|_{p,1}+\|\mathrm{d}u\|_p\|\xi\|_{q_{i+1}}\big). \tag{4.16}$$

The recursion (4.15) implies that

$$q_{i+1}=\frac{2p}{2p+(p-2)q_i}q_i \quad\Longrightarrow\quad \text{if } q_i>0, \text{ then } 0<q_{i+1}<q_i.$$

Thus, if $q_i>2$ for all i, then the sequence $\{q_i\}$ must have a limit $q\ge 2$ with

$$q=\frac{2p}{2p+(p-2)q}q \quad\Longrightarrow\quad (p-2)q=0 \quad\Longrightarrow\quad q=0,$$

since $p>2$ by assumption. Thus, $q_N\le p$ for N sufficiently large and the first claim follows from (4.13) and the equations (4.16) with i running from 1 to N, where N is the smallest integer such that $q_{N+1}\le p$. The second claim follows immediately from the first. $\square$

4.3 Elliptic estimates

If $A_1=B_{R_1,r_1}$ and $A_2=\bar{B}_{R_2,r_2}$ are two annuli in $\mathbb{R}^2$, we write $A_2\Subset_\delta A_1$ if $R_1-R_2>\delta$ and $r_2-r_1\ge\delta$.

Lemma 4.11. *For any $\delta>0$, $p\ge 1$, and open annulus A_1, there exists $C_{\delta,p}(A_1)>0$ such that for any annulus $A_2\Subset_\delta A_1$ and $\xi\in C^\infty(A_1;\mathbb{C}^k)$,*

$$\big\|\xi|_{A_2}\big\|_{p,1}\le C_{\delta,p}(A_1)\big(\|\bar\partial\xi\|_p+\|\mathrm{d}\xi\|_2+\|\xi\|_1\big),$$

where the norms are taken with respect to the standard metric on $\mathbb{R}^2$.

Proof. We can assume that A_2 is the maximal annulus such that $A_2 \Subset_\delta A_1$. Let $\eta\colon A_1 \longrightarrow [0,1]$ be a compactly supported smooth function such that $\eta|_{A_2}=1$. By the fundamental elliptic inequality for the $\bar\partial$-operator on S^2 [4, Lemma C.2.1],

$$\begin{aligned}\big\|\xi|_{A_2}\big\|_{p,1} &\le \|\eta\xi\|_{p,1} \le C_p(A_1)\big(\|\bar\partial(\eta\xi)\|_p+\|\eta\xi\|_p\big)\\ &\le C_p(A_1)\big(\|\bar\partial\xi\|_p+\|(\mathrm{d}\eta)\xi\|_p+\|\eta\xi\|_p\big).\end{aligned} \tag{4.17}$$

By Corollary 4.5 with $(p,q)=(2,p)$ and $(p,q)=(1,2)$ and Hölder's inequality,

$$\begin{aligned}\|\eta\xi\|_p &\le C_p(A_1)\|\mathrm{d}(\eta\xi)\|_2 \le C_p(A_1)\big(\|\mathrm{d}\xi\|_2+\|(\mathrm{d}\eta)\xi\|_2\big)\\ &\le \widetilde{C}_p(A_1)\big(\|\mathrm{d}\xi\|_2+\|\mathrm{d}((\mathrm{d}\eta)\xi)\|_1\big) \le \widetilde{C}_{p,\delta}(A_1)\big(\|\mathrm{d}\xi\|_2+\|\mathrm{d}\xi\|_1+\|\xi\|_1\big)\\ &\le C_{\delta,p}(A_1)\big(\|\mathrm{d}\xi\|_2+\|\xi\|_1\big).\end{aligned} \tag{4.18}$$

Similarly,

$$\|(\mathrm{d}\eta)\xi\|_p \le C_{\delta,p}(A_1)\big(\|\mathrm{d}\xi\|_2+\|\xi\|_1\big). \tag{4.19}$$

The claim follows by plugging (4.18) and (4.19) into (4.17). $\square$

Corollary 4.12. *For any $\delta>0$, $p\ge1$, and open annulus A_1, there exists $C_{\delta,p}(A_1)>0$ such that for any annulus $A_2\Subset_\delta A_1$, and $\xi\in C^\infty(A_1;\mathbb{C}^n)$,*

$$\|\mathrm{d}\xi|_{A_2}\|_p \le C_{\delta,p}(A_1)\big(\|\bar\partial\xi\|_p+\|\mathrm{d}\xi\|_2\big).$$

Proof. With $|A_1|$ denoting the area of A_1, let

$$\bar\xi=\frac{1}{|A_1|}\int_{A_1}\xi$$

be the average value of ξ. By Lemma 4.11,

$$\begin{aligned}\|\mathrm{d}\xi|_{A_2}\|_p=\|\mathrm{d}(\xi-\bar\xi)|_{A_2}\|_p &\le C_{\delta,p}(A_1)\big(\|\bar\partial(\xi-\bar\xi)\|_p+\|\mathrm{d}(\xi-\bar\xi)\|_2+\|\xi-\bar\xi\|_1\big)\\ &= C_{\delta,p}(A_1)\big(\|\bar\partial\xi\|_p+\|\mathrm{d}\xi\|_2+\|\xi-\bar\xi\|_1\big).\end{aligned} \tag{4.20}$$

The claim follows by applying Corollary 4.7 with $\zeta=\xi-\bar\xi$. $\square$

Remark 4.13. The case $r_1>0$ (which is the case needed for gluing pseudo-holomorphic maps in symplectic topology) follows from Corollary 3.7; Corollary 4.7 can be used to obtain a sharper statement in this case (that $C_{\delta,p}(A_1)$ does not depend on r_1). The $r_1=0$ case requires only the first two steps in the proof of Corollary 4.7.

A smooth generalized CR-operator in a smooth complex vector bundle (E,∇) with connection over an almost complex manifold (M,J) is an operator of the form

$$D=\bar\partial_\nabla+A\colon \Gamma(M;E)\longrightarrow \Gamma(M;T^*M^{0,1}\otimes_{\mathbb{C}}E),$$

where

$$\bar\partial_\nabla\xi=\frac12\big(\nabla\xi+\mathfrak{i}\nabla_J\xi\big)\quad \forall\,\xi\in\Gamma(M;TM),\qquad A\in\Gamma\big(M;\mathrm{Hom}(E;T^*M^{0,1}\otimes_{\mathbb{C}}E)\big).$$

If in addition $u\colon\Sigma\longrightarrow M$ is a smooth map from an almost complex manifold $(\Sigma,\mathfrak{j})$, the pull-back CR-operator is given by

$$D_u=\bar\partial_{\nabla^u}+A\circ\partial u\colon \Gamma(u;E)\longrightarrow \Gamma^{0,1}(u;E).$$

Proposition 4.14. *If (M,g) is a Riemannian manifold with an almost complex structure J, $(E,\langle,\rangle,\nabla)$ is a normed complex vector bundle with connection over M and a smooth generalized CR-operator D, and $p\geq 1$, then for every compact subset $K\subset M$, $\delta>0$, and open annulus $A_1\subset\mathbb{R}^2$, there exists $C_{K;\delta,p}(A_1)\in\mathbb{R}^+$ with the following property. If $u\in C^\infty(A_1;M)$ is such that $\operatorname{Im} u\subset K$, $\xi\in\Gamma(u;E)$, and $A_2\Subset_\delta A_1$ is an annulus, then*

$$\big\|\nabla^u\xi|_{A_2}\big\|_p \le C_{K;\delta,p}(A_1)\big(\|D_u\xi\|_p+\|\nabla^u\xi\|_2+\|\xi\otimes\mathrm{d}u\|_p\big),$$

where the norms are taken with respect to the standard metric on $\mathbb{R}^2$.

Proof. We continue with the setup in the proof of Lemma 4.8. By Corollary 4.12,

$$\begin{aligned}\big\|\mathrm{d}\widetilde{\xi}_i|_{A_2}\big\|_p &\le C_{i;\delta,p}(A_1)\big(\|\bar\partial\widetilde{\xi}_i\|_p+\|\mathrm{d}\widetilde{\xi}_i\|_2\big)\\ &\le C'_{i;\delta,p}(A_1)\big(\big\|\bar\partial\xi_i|_{u^{-1}(U_i)}\big\|_p+\big\|\mathrm{d}\xi_i|_{u^{-1}(U_i)}\big\|_2+\|\xi\otimes\mathrm{d}u\|_p\big).\end{aligned}\tag{4.21}$$

Since ∇ commutes with the complex structure in E and $\widetilde{\xi}_i=\xi_i$ on $u^{-1}(W_i)$, it follows from (4.12) and (4.21) that

$$\begin{aligned}\big\|\nabla^u\xi|_{A_2\cap u^{-1}(W_i)}\big\|_p &\le \big\|\mathrm{d}\widetilde{\xi}_i|_{A_2}\big\|_p+C_K\|\xi\otimes\mathrm{d}u\|_p\\ &\le \widetilde{C}_{i;\delta,p}(A_1)\big(\|\bar\partial_{\nabla^u}\xi\|_p+\|\nabla^u\xi\|_2+\|\xi\otimes\mathrm{d}u\|_p\big)\\ &\le \widetilde{C}'_{i;\delta,p}(A_1)\big(\|D_u\xi\|_p+\|\nabla^u\xi\|_2+\|\xi\otimes\mathrm{d}u\|_p\big).\end{aligned}\tag{4.22}$$

The claim is obtained by summing the last equation over all i. □

Lemma 4.15. *If (M,g) is a Riemannian manifold with an almost complex structure J, $(E,\langle,\rangle,\nabla)$ is a normed complex vector bundle with connection over M and a smooth generalized CR-operator D, and $p>2$, then for every compact subset $K\subset M$ and open ball $B\subset\mathbb{R}^2$, there exists $C_{K;B,p}\in C^\infty(\mathbb{R};\mathbb{R})$ with the following property. If $u\in C^\infty(B;M)$ is such that $\operatorname{Im} u\subset K$ and $\xi\in\Gamma_c(u;E)$, then*

$$\|\xi\|_{p,1}\le C_{K;B,p}(\|\mathrm{d}u\|_p)\big(\|D_u\xi\|_p+\|\xi\|_p\big),$$

where the norms are taken with respect to the standard metric on $\mathbb{R}^2$.

Proof. By an argument nearly identical to the proof of Proposition 4.14,

$$\|\xi\|_{p',1}\le C_{K;p'}(B)\big(\|D_u\xi\|_{p'}+\|\xi\|_{p'}+\|\xi\otimes\mathrm{d}u\|_{p'}\big)$$

for any $p'\geq 1$. On the other hand, by Proposition 4.10,

$$\|\xi\|_{C^0}\le C_{K;B,\widetilde{p}}(\|\mathrm{d}u\|_{\widetilde{p}})\|\xi\|_{\widetilde{p},1},$$

where $\widetilde{p}=(p+2)/2$. Proceeding as in the proof of Proposition 4.10, we then obtain

$$\begin{aligned}\|\xi\|_{p,1}&\le C_{K;B,p}(\|\mathrm{d}u\|_{\widetilde{p}})\big(\|D_u\xi\|_p+\|\xi\|_p+\|\mathrm{d}u\|_p\|\xi\|_{\widetilde{p},1}\big),\\ \|\xi\|_{\widetilde{p},1}&\le C_{K;\widetilde{p}}(B)\big(\|D_u\xi\|_p+\|\xi\|_p+\|\mathrm{d}u\|_p\|\xi\|_{q_1}\big),\\ \|\xi\|_{q_i}&\le C_{K;p_i,q_i}(B)\big(\|\xi\|_{p_i,1}+\|\xi\otimes\mathrm{d}u\|_{p_i}\big)\\ &\le C_{K;B,i}(\|\mathrm{d}u\|_p)\big(\|D_u\xi\|_p+\|\xi\|_p+\|\mathrm{d}u\|_p\|\xi\|_{q_{i+1}}\big);\end{aligned}$$

we stop the recursion at the same value of $i=N$ as in the proof of Proposition 4.10. □

Proposition 4.16. *If (M, g) is a Riemannian manifold with an almost complex structure J, $(E, \langle,\rangle, \nabla)$ is a normed complex vector bundle with connection over M and a smooth generalized CR-operator D, and $p > 2$, then for every compact subset $K \subset M$ and compact Riemann surface (Σ, g_Σ), there exists $C_{K;\Sigma,p} \in C^\infty(\mathbb{R};\mathbb{R})$ with the following property. If $u \in C^\infty(\Sigma; M)$ is such that $\mathrm{Im}\, u \subset K$ and $\xi \in \Gamma(u; E)$, then*

$$\|\xi\|_{p,1} \leq C_{K;\Sigma,p}\big(\|\mathrm{d}u\|_p\big)\big(\|D_u\xi\|_p + \|\xi\|_p\big).$$

Proof. This statement is immediate from Lemma 4.15. □

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