MAT 531 Geometry/Topology Homework 6

1. Consider a complex polynomial \( f : \mathbb{C} \to \mathbb{C} \). Prove that it has only finitely many critical values. Deduce that the mapping degree of \( f \) is independent of the choice of a regular value.

2. Prove the Fundamental Theorem of Algebra: any nonconstant complex polynomial has at least one complex root. Find the mapping degree of a complex polynomial in terms of its algebraic degree.

3. Let \( X \) be a smooth manifold of dimension \( m \) and \( Y \) a smooth manifold of dimension \( n \leq m \). Consider a smooth map \( f : X \to Y \). A point \( x \in X \) is called a critical point of \( f \) if the differential \( df_x \) is not onto (i.e., it does not have the maximal rank). The image of any critical point is called a critical value. A regular value of \( f \) is any point \( y \in Y \) that is not a critical value. Prove that for any regular value \( y \in Y \), the subset \( f^{-1}(y) \subseteq X \) is a smooth submanifold of dimension \( m - n \). Hint: use the Implicit Function Theorem.

4. Let \( X \) and \( Y \) be smooth manifolds. A smooth homotopy between two smooth maps \( f, g : X \to Y \) is defined as a smooth map \( F : X \times [0, 1] \to Y \) such that \( F(x, 0) = f(x) \) and \( F(x, 1) = g(x) \) for all \( x \in X \). Suppose that both \( X \) and \( Y \) are oriented, and that smooth maps \( f, g : X \to Y \) are smoothly homotopic. Prove that \( f \) and \( g \) have the same mapping degree. You can use the following fact without proof: there exists a point \( y \in Y \) that is a regular value of \( F, f \) and \( g \).

5. Let \( f, g : X \to Y \) be two diffeomorphisms of a smooth manifold \( X \) to a smooth manifold \( Y \). A smooth homotopy \( F \) connecting \( f \) with \( g \) is called a smooth isotopy if the map \( F(\cdot, t) : x \in X \mapsto F(x, t) \in Y \) is a diffeomorphism for each \( t \in [0, 1] \). Prove that the time 1 flow \( \phi^1_t : X \to X \) of any smooth vector field \( v \) on \( X \) is smoothly isotopic to the identity map.

6*. Let \( X \) be a connected smooth manifold. Prove that for any pair of points \( y_1, y_2 \in Y \), there exists a smooth self-map \( h : Y \to Y \) smoothly isotopic to the identity and such that \( h(y_1) = y_2 \). Deduce that the mapping degree of a smooth map \( f : X \to Y \) does not depend on the choice of a regular value in \( Y \), i.e. \( \text{mdeg}_{y_1}(f) = \text{mdeg}_{y_2}(f) \).

\( \text{Hint:} \) define a smooth vector field on \( X \), whose time 1 flow maps \( x_1 \) to \( x_2 \). It may be convenient first to define this vector field on a neighborhood of the path connecting \( x_1 \) to \( x_2 \).