MAT 531 Geometry/Topology Homework 2

A vector field on a smooth manifold is said to be smooth, if it is smooth in any local chart of this manifold.

1. Let $v$ be a smooth vector field on a smooth manifold $X$. Prove that for any smooth function $f$ on $X$, the function $\partial_v f$ is also smooth.

2. Let $\phi_t^v$ be the flow (for time $t$) of a smooth vector field $v$ on an open subset of $\mathbb{R}^n$. Prove that

$$\phi_t^v(x_0) = x_0 + \epsilon v(x_0) + \frac{\epsilon^2}{2} d_{x_0}v[v(x_0)] + o(\epsilon^2).$$

Here $d_{x_0}v[v(x_0)]$ means the image of $v(x_0)$ under the action of the linear operator $d_{x_0}v$. In coordinates,

$$\phi_t^v(x_0)^i = x_0^i + \epsilon v^i(x_0) + \frac{\epsilon^2}{2} \frac{\partial v^i}{\partial x^j} v^j(x_0) + o(\epsilon^2).$$

3. For any pair of smooth vector fields $v$ and $w$ on an open subset of $\mathbb{R}^n$, prove that

$$\lim_{(\epsilon, \delta) \to 0} \frac{\phi_t^v \circ \phi_t^w(x) - \phi_t^w \circ \phi_t^v(x)}{\epsilon \delta} = d_x v[w] - d_x v[w] = [v, w](x).$$

4. Compute the commutator $[x \partial_x + y \partial_y, x \partial_y - y \partial_x]$. Sketch the corresponding vector fields in the plane.

5. Introduce the spherical coordinates $(r, \phi, \psi)$ in $\mathbb{R}^3$:

$$x = r \cos(\phi) \cos(\psi), \quad y = r \sin(\phi) \cos(\psi), \quad z = r \sin(\psi).$$

Express the vector fields $\partial_r$, $\partial_\phi$ and $\partial_\psi$ in spherical coordinates.

6. Let $f : X \to Y$ be a diffeomorphism of a smooth manifold $X$ to a smooth manifold $Y$. Prove that

$$f_*(v, w) = [f_*(v), f_*(w)]$$

for any pair of smooth vector fields $v$ and $w$ on $X$. Here $f_*(v)$ is a vector field on $Y$, which is defined as follows. For any point $q \in Y$, the tangent vector $f_*(v)_q$ at $q$ is the image of the tangent vector $v_p$ at $p = f^{-1}(q) \in X$ under the linear map $d_pf : T_pX \to T_qY$. 

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