MAT531 Geometry/Topology Final Exam

1. Let X be the subset of $\mathbb{R}^2$ defined by the equation $xy = 0$. Is X a smooth submanifold of $\mathbb{R}^2$? As a topological space, does it have a smooth structure?

2. Consider a smooth action of $\mathbb{R}^2$ on a smooth manifold X (a *smooth action* means a smooth map assigning a diffeomorphism $\rho^a : X \to X$ to each vector $a \in \mathbb{R}^2$ so that $\rho^{a+b} = \rho^a \circ \rho^b$ for $a, b \in \mathbb{R}^2$). Prove that ρ^a is a time-1 flow of some smooth vector field v^a for any $a \in \mathbb{R}^2$. Prove that $[v^a, v^b] = 0$ for $a, b \in \mathbb{R}^2$. *Hint:* $v^a_x = \frac{d}{dt}\rho^{ta}(x)$.

3. Let f be a smooth function on $\mathbb{R}^2$. Suppose that df does not vanish on the subset $\{f = 0\}$, so that this subset is a smooth submanifold. Prove that, in a neighborhood of $\{f = 0\}$, there exists a smooth 1-form α such that $\alpha \wedge df = dx \wedge dy$. The restriction of α to $\{f = 0\}$ is well-defined (i.e. independent of the choice of α).

4. (a) Give an example of a closed 1-form on $\mathbb{R}^2 - 0$ that is not exact. Justify your answer.
(b)* Show that $H^2(\mathbb{R}^3 - 0) \neq 0$. *Hint:* there is a natural projection $\mathbb{R}^3 - 0 \to S^2$.

5. Prove that there is no smooth surface Σ in $\mathbb{R}^3$ that is tangent to the vector fields $v_1 = -z\partial_x + \partial_y$, $v_2 = -xy\partial_x + \partial_z$ at each point of Σ.

6*. Let γ be a loop in $\mathbb{R}^2$ such that $\int_\gamma xy(xdx - ydy) = 0$. Prove that γ has a point of self-intersection.