MAT 531: Topology&Geometry, II
Spring 2006

Problem Set 5
Due on Thursday, 3/2, in class

Note: This problem set has two pages.

1. Suppose M is a smooth n-manifold.
(a) Let α be a nowhere-zero n-form on M. Show that for every $m \in M$ there exists a chart $(x_1, \ldots, x_n)\colon \mathcal{U} \longrightarrow \mathbb{R}^n$ near m such that
$$\alpha|_{\mathcal{U}} = dx_1 \wedge \ldots \wedge dx_n.$$
(b) Let α be a nowhere-zero closed $(n-1)$-form on M. Show that for every $m \in M$ there exists a chart $(x_1, \ldots, x_n)\colon \mathcal{U} \longrightarrow \mathbb{R}^n$ near m such that
$$\alpha|_{\mathcal{U}} = dx_2 \wedge dx_3 \wedge \ldots \wedge dx_n.$$

2. Chapter 2, #5 (p78)

3. A two-form ω on a smooth manifold M is called `symplectic` if ω is closed (i.e. $d\omega = 0$) and everywhere nondegenerate[1]. Suppose ω is a symplectic form on M.
(a) Show that the dimension of M is even and the map
$$TM \longrightarrow T^*M, \qquad X \longrightarrow i_X\omega,$$
is a vector bundle isomorphism ($i_X\omega$ is the contraction w.r.t. X, i.e. the dual of $X\wedge$).
(b) If $H : M \longrightarrow \mathbb{R}$ is a smooth map, let $X_H \in \Gamma(M; TM)$ be the preimage of dH under this isomorphism. Assume that X_H is a complete vector field, so that the flow
$$\varphi\colon \mathbb{R}\times M \longrightarrow M, \qquad (t, m) \longrightarrow \varphi_t(m),$$
is globally defined. Show that for every $t \in \mathbb{R}$, the time-t flow $\varphi_t\colon M \longrightarrow M$ is a symplectomorphism, i.e. $\varphi_t^*\omega = \omega$.
Note: In such a situation, H is called a `Hamiltonian` and φ_t a `Hamiltonian symplectomorphism`.

4. Suppose X and Y are smooth vector fields on a manifold M. Show that for every $m \in M$ and $f \in C^\infty(M)$,
$$\lim_{s,t \longrightarrow 0} \frac{f\big(Y_{-s}(X_{-t}\big(Y_s(X_t(m))\big))\big) - f(m)}{s\,t} = [X, Y]_m f \in \mathbb{R}.$$
Do not forget to explain why the limit exists.
Note: This means that the extent to which the flows of X and Y do not commute (i.e. the rate of change in the "difference" between $Y_s\circ X_t$ and $X_t\circ Y_s$) is measured by $[X, Y]$.

[1] This means that $\omega_m \in \Lambda^2 T_m^*M$ is nondegenerate for every $m \in M$, i.e. for every $v \in T_mM$ such that $v \neq 0$ there exists $w \in T_mM$ such that $\omega(v, w) \neq 0$.

5. Suppose M is a 3-manifold, α is a nowhere-zero one-form on M, and $m \in M$. Show that
   (a) if there exists an embedded 2-dimensional submanifold $P \subset M$ such that $m \in P$ and $\alpha|_{TP} = 0$, then $(\alpha \wedge d\alpha)|_m = 0$.
   (b) if there exists a neighborhood $\mathcal{U}$ of m in M such that $(\alpha \wedge d\alpha)|_{\mathcal{U}} = 0$, then there exists an embedded 2-dimensional submanifold $P \subset M$ such that $m \in P$ and $\alpha|_{TP} = 0$.

   Note: If the top form $\alpha \wedge d\alpha$ on M is nowhere-zero, α is called a `contact form`. In this case, it has no integrable submanifolds at all.