MAT 531: Topology&Geometry, II
Spring 2006

Problem Set 4
Due on Thursday, 2/23, in class

Please read the updated version of *Notes on Vector Bundles* before starting on Problem 4.

1. Chapter 1, #13ad (p51)

2. Let U and V be the vector fields on $\mathbb{R}^3$ given by

$$U(x,y,z)=\frac{\partial}{\partial x} \qquad \text{and} \qquad V(x,y,z)=F(x,y,z)\frac{\partial}{\partial y}+G(x,y,z)\frac{\partial}{\partial z},$$

   where F and G are smooth functions on $\mathbb{R}^3$. Show that there exists a smooth 2-dimensional foliation $\mathcal{F}$ on $\mathbb{R}^3$ such that the vector fields U and V are everywhere tangent to $\mathcal{F}$[1] if and only if

$$F(x,y,z)=f(y,z)\,e^{h(x,y,z)} \qquad \text{and} \qquad G(x,y,z)=g(y,z)\,e^{h(x,y,z)}$$

   for some $f,g\in C^\infty(\mathbb{R}^2)$ and $h\in C^\infty(\mathbb{R}^3)$ such that (f,g) does not vanish on $\mathbb{R}^2$.

3. Chapter 2, #13 (p79)

4. Let $\Lambda^n_{\mathbb{C}}T\mathbb{C}P^n\longrightarrow\mathbb{C}P^n$ be the top exterior power of the vector bundle $T\mathbb{C}P^n$ taken over $\mathbb{C}$. Show that $\Lambda^n_{\mathbb{C}}T\mathbb{C}P^n$ is isomorphic to the line bundle

$$\gamma_n^{*\,\otimes(n+1)}\equiv\underbrace{\gamma_n^*\otimes\ldots\otimes\gamma_n^*}_{n+1},$$

   where $\gamma_n\longrightarrow\mathbb{C}P^n$ is the tautological line bundle (isomorphic as complex line bundles).
   Hint: There are a number of ways of doing this, including:
   (i) construct an isomorphism between the two line bundles;
   (ii) use Problems 4 and 5 from PS1 to determine transition data for $\Lambda^n_{\mathbb{C}}T\mathbb{C}P^n$ and $\gamma_k^{*\,\otimes(n+1)}$. However, you will need to modify trivializations for one of the line bundles to arrive at the same transition data.
   (iii) show that there exists a short exact sequence of vector bundles

$$0\longrightarrow\mathbb{C}P^n\times\mathbb{C}\longrightarrow(n+1)\gamma_n^*\longrightarrow T\mathbb{C}P^n\longrightarrow 0$$

   and this implies the claim (exact means that at each position the kernel of the outgoing map equals to the image of the incoming map over every point of M.)

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[1] This means that $\mathcal{F}$ is a collection of *embedded* submanifolds of $\mathbb{R}^3$ that partition $\mathbb{R}^3$ such that the tangent bundles of the submanifolds are spanned by U and V.