MAT 531: Topology&Geometry, II
Spring 2011

Midterm Solutions

Problem 1 (15pts)

Suppose M is a smooth manifold, $X,Y\in\Gamma(M;TM)$ are smooth vector fields on M, and $g\in C^\infty(M)$ is a smooth function on M. Show directly from the definition that

$$[gX,Y]=g[X,Y]-Y(g)X\,.$$

(You can assume that $[\cdot,\cdot]$ is whatever object it is supposed to be, but do state what you are taking it to be.)

By definition, the Lie bracket $[X,Y]$ of two smooth vector fields X and Y is another smooth vector field on M, i.e. an element of $\Gamma(M;TM)$. In particular,

$$[gX,Y],\, g[X,Y]-Y(g)X\colon C^\infty(M)\longrightarrow C^\infty(M)$$

are linear maps. They are given by

$$\begin{aligned}
[gX,Y]f \equiv \{gX\}(Yf)-Y(gXf) &= \{gX\}(Yf)-\big((Yg)(Xf)+gY(Xf)\big)\\
&= g\cdot\big(X(Yf)-Y(Xf)\big)-(Yg)(Xf)\\
\big\{g[X,Y]-Y(g)X\big\}(f)\equiv g[X,Y](f)-Y(g)X(f) &= g\cdot\big(X(Yf)-Y(Xf)\big)-(Yg)(Xf).
\end{aligned}$$

The second equality on the first line holds because Y is a derivation on $C^\infty(M)$. Since the two vector fields act in the same way on all functions, they are equal.

Problem 2 (20pts)

Let $f\colon M\longrightarrow N$ be a smooth surjective map.

(a) *Suppose f is a submersion (d_pf is onto for all $p\in M$). Show that a map $h\colon N\longrightarrow\mathbb{R}$ is smooth if and only if the map $h\circ f: M\longrightarrow\mathbb{R}$ is smooth.*

(b) *Which of the two implications can fail if f is not assumed to be a submersion? Give an example.*

(a) If $h\colon N\longrightarrow\mathbb{R}$ is a smooth map, then so is $h\circ f: M\longrightarrow\mathbb{R}$, because a composition of smooth maps is smooth. Suppose $h\circ f: M\longrightarrow\mathbb{R}$ is a smooth map and $q\in N$. Choose any point $p\in f^{-1}(q)$; such a point exists because f is surjective. By the local structure of submersions, there exist smooth charts $\varphi\colon U\longrightarrow\mathbb{R}^m=\mathbb{R}^n\times\mathbb{R}^k$ around p and $\psi\colon V\longrightarrow\mathbb{R}^n$ around q so that $\pi_1\circ\varphi=\psi\circ f$, where $\pi_1\colon\mathbb{R}^n\times\mathbb{R}^k\longrightarrow\mathbb{R}^n$ is the projection map. Thus,

$$h\circ\psi^{-1}=h\circ\psi^{-1}\circ\pi_1\big|_{\mathbb{R}^n\times0}=h\circ f\circ\varphi^{-1}\big|_{\mathbb{R}^n\times0}\colon\mathbb{R}^n\times0\longrightarrow\mathbb{R}\,.$$

Since $h \circ f \circ \varphi^{-1} : \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}$ is a smooth map (because $h \circ f : M \longrightarrow \mathbb{R}$ is smooth and $\varphi : U \longrightarrow \mathbb{R}^m$ is a smooth chart), its restriction to the smooth submanifold $\mathbb{R}^n \times 0$ is smooth. Thus, h is a smooth map on N (since for every point $q \in N$, there exists a smooth chart $\psi : V \longrightarrow \mathbb{R}^n$ so that $h \circ \psi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is smooth).

(b) If $h : N \longrightarrow \mathbb{R}$ is a smooth map, then so is $h \circ f : M \longrightarrow \mathbb{R}$, whether or not f is a submersion (because a composition of smooth maps is smooth). However, the converse need not hold if h is not a submersion. For example, let $M = N = \mathbb{R}$ (with the standard smooth structure),

$$f : M \longrightarrow N, \quad t \longrightarrow t^3, \qquad h : N \longrightarrow \mathbb{R}, \quad t \longrightarrow t^{1/3}.$$

The maps $f : M \longrightarrow N$ and $h \circ f = \mathrm{id}_{\mathbb{R}} : M \longrightarrow \mathbb{R}$ are then smooth, f is surjective, but $h : N \longrightarrow \mathbb{R}$ is not smooth.

Problem 3 (20pts)

Let $\alpha = \mathrm{d}x_1 + f\mathrm{d}x_2$ be a smooth 1-form on $\mathbb{R}^3$ (so $f \in C^\infty(\mathbb{R}^3)$). Show that for every $p \in \mathbb{R}^3$ there exists a diffeomorphism

$$\varphi = (y_1, y_2, y_3) : U \longrightarrow V$$

from a neighborhood U of p to an open subset V of $\mathbb{R}^3$ such that $\alpha|_U = \mathrm{d}y_1$ if and only if f does not depend on x_1 or x_3 (depends on x_2 only).

If $\varphi = (y_1, y_2, y_3) : U \longrightarrow V$ is a chart on $\mathbb{R}^3$ such that $\alpha|_U = \mathrm{d}y_1$, then

$$(\mathrm{d}\alpha)|_U = \mathrm{d}\big(\alpha|_U\big) = \mathrm{d}\big(\mathrm{d}y_1\big) = \mathrm{d}^2 y_1 = 0.$$

Thus, if for every $p \in \mathbb{R}^3$ there exists a diffeomorphism $\varphi = (y_1, y_2, y_3) : U \longrightarrow V$ from a neighborhood U of p to an open subset V of $\mathbb{R}^3$ such that $\alpha|_U = \mathrm{d}y_1$, then

$$0 = \mathrm{d}\alpha = \mathrm{d}f \wedge \mathrm{d}x_2 = \frac{\partial f}{\partial x_1}\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \frac{\partial f}{\partial x_3}\mathrm{d}x_3 \wedge \mathrm{d}x_2.$$

Thus, the partials of f with respect to x_1 and x_3 vanish identically and so f depends only on x_2.

Conversely, suppose $f = f(x_2)$ depends only on x_2. Let $F(x) = \displaystyle\int_0^x f(t)\mathrm{d}t$, so that $\alpha = \mathrm{d}(x_1 + F(x_2))$. Define

$$\varphi = (y_1, y_2, y_3) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad \text{by} \qquad y_1 = x_1 + F(x_2), \quad y_2 = x_2, \quad y_3 = x_3.$$

This is a smooth bijective map. Its Jacobian,

$$\begin{pmatrix} 1 & f(x_2) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is everywhere invertible. Thus, φ is a required diffeomorphism that works for all $p \in \mathbb{R}^3$.

The Frobenius Theorem can also be used to obtain the latter implication. Since α is nowhere zero, it determines a line subbundle $W\subset T^*\mathbb{R}^3$; any section of this subbundle has the form $h\alpha$ for some $h\in C^\infty(M)$. Since $\mathrm{d}\alpha=0$ (under the assumption that g depends only on x_2),

$$\mathrm{d}\big(h\alpha\big) = \mathrm{d}h\wedge\alpha + h\mathrm{d}\alpha = \mathrm{d}h\wedge\alpha \qquad\Longrightarrow\qquad \mathrm{d}\colon \Gamma(\mathbb{R}^3;W)\longrightarrow \Gamma(\mathbb{R}^3;W\wedge T^*\mathbb{R}^3)\subset E^2(\mathbb{R}^3).$$

So, the assumption in the second, differential-form, version of Frobenius Theorem is satisfied. Thus, for every $p\in\mathbb{R}^3$ there exists a smooth chart $\varphi=(y_1,y_2,y_3)\colon U\longrightarrow\mathbb{R}^3$ from a neighborhood U of p to an open subset V of $\mathbb{R}^3$ such that the tangent spaces to the vertical slices $\varphi^{-1}(y_1\times\mathbb{R}^2)$ are the kernels of α. Thus, $\alpha|_U=g\mathrm{d}y_1$ for some $g\in C^\infty(U)$, as the tangent spaces to the slices are spanned by the coordinate vectors $\partial/\partial y_2$ and $\partial/\partial y_3$. Since $\mathrm{d}\alpha=0$,

$$0=\mathrm{d}\big(\alpha|_U\big)=\mathrm{d}(g\mathrm{d}y_1)=\frac{\partial g}{\partial y_2}\mathrm{d}y_2\wedge\mathrm{d}y_1+\frac{\partial g}{\partial y_3}\mathrm{d}y_3\wedge\mathrm{d}y_1.$$

Thus, the partials of $g\circ\varphi^{-1}$ with respect to y_2 and y_3 vanish identically and so

$$g=h\circ y_1:\mathbb{R}^3\longrightarrow\mathbb{R}$$

for some $h\in C^\infty(\mathbb{R})$. Define $H:\mathbb{R}\longrightarrow\mathbb{R}$ by $H(y)=\displaystyle\int_0^y h(t)\mathrm{d}t$ so that $\alpha|_U=y_1^*\mathrm{d}H$. Since α is nowhere 0, h is nowhere 0 and thus $H\colon\mathbb{R}\longrightarrow\mathbb{R}$ is a diffeomorphism onto an open subset of $\mathbb{R}$. It follows that

$$\psi=(z_1,z_2,z_3)\equiv(H\circ y_1,y_2,y_3)\colon U\longrightarrow\mathbb{R}^3$$

is a smooth chart p such that $\alpha|_U=\mathrm{d}z_1$.

Problem 4 (20pts)

Let M and N be smooth nonempty manifolds and $\pi_1:M\times N\longrightarrow M$ and $\pi_2:M\times N\longrightarrow N$ the projection maps. Show directly from the definitions that the homomorphism

$$\Phi\colon H^1_{deR}(M)\oplus H^1_{deR}(N)\longrightarrow H^1_{deR}(M\times N),\qquad \big([\alpha],[\beta]\big)\longrightarrow\big[\pi_1^*\alpha+\pi_2^*\beta\big],$$

is well-defined and injective.

If $\alpha\in E^1(M)$, $\beta\in E^1(N)$, $\mathrm{d}\alpha=0$, and $\mathrm{d}\beta=0$, then

$$\mathrm{d}\big(\pi_1^*\alpha+\pi_2^*\beta\big)=\mathrm{d}\pi_1^*\alpha+\mathrm{d}\pi_2^*\beta=\pi_1^*\mathrm{d}\alpha+\pi_2^*\mathrm{d}\beta=\pi_1^*0+\pi_2^*0=0;$$

thus, the one-form $\pi_1^*\alpha+\pi_2^*\beta$ on $M\times N$ is closed and so determines a class in $H^1_{deR}(M\times N)$. If in addition

$$\alpha'\in E^1(M),\quad \beta'\in E^1(N),\quad \mathrm{d}\alpha'=0,\quad \mathrm{d}\beta'=0,\quad [\alpha]=[\alpha']\in H^1_{deR}(M),\quad [\beta]=[\beta']\in H^1_{deR}(N),$$

then $\alpha'=\alpha+\mathrm{d}f$ for some $f\in C^\infty(M)$ and $\beta'=\beta+\mathrm{d}g$ for some $g\in C^\infty(N)$. Thus,

$$\pi_1^*\alpha'+\pi_2^*\beta'=\pi_1^*\alpha+\pi_1^*\mathrm{d}f+\pi_2^*\beta+\pi_2^*\mathrm{d}g=\pi_1^*\alpha+\pi_2^*\beta+\mathrm{d}\big(\pi_1^*f+\pi_2^*g\big).$$

So $\big[\pi_1^*\alpha'+\pi_2^*\beta'\big]=[\pi_1^*\alpha+\pi_2^*\beta]\in H^1_{deR}(M\times N)$, i.e. the map Φ is well-defined. It is a homomorphism because the map

$$\Phi\colon E^1(M)\oplus E^1(N)\longrightarrow E^1(M\times N),\qquad (\alpha,\beta)\longrightarrow \pi_1^*\alpha+\pi_2^*\beta,$$

is.

It remains to show that this homomorphism is injective. Let $p\in M$ and $q\in N$ be any points and

$$\iota_p\colon N\longrightarrow M\times N,\quad y\longrightarrow (p,y),\qquad \iota_q\colon M\longrightarrow M\times N,\quad x\longrightarrow (x,q),$$

inclusions of N and M as vertical and horizontal slices, respectively. In particular,

$$\pi_1\circ\iota_q=\mathrm{id}_M,\qquad \pi_2\circ\iota_p=\mathrm{id}_N\,,$$

while $\pi_1\circ\iota_p\colon N\longrightarrow M$ and $\pi_2\circ\iota_q\colon M\longrightarrow N$ are constant maps. Suppose $\alpha\in E^1(M)$, $\beta\in E^1(N)$, $\mathrm{d}\alpha=0$, $\mathrm{d}\beta=0$, and

$$\big[\pi_1^*\alpha+\pi_2^*\beta\big]=0\in H^1_{deR}(M\times N).$$

Then, there exists $h\in C^\infty(M\times N)$ such that

$$\begin{aligned}\pi_1^*\alpha+\pi_2^*\beta=\mathrm{d}h \implies\ & \mathrm{d}(h\circ\iota_q)=\iota_q^*\mathrm{d}h=\iota_q^*\pi_1^*\alpha+\iota_q^*\pi_2^*\beta=(\pi_1\circ\iota_q)^*\alpha+(\pi_2\circ\iota_q)^*\beta=\mathrm{id}_M^*\alpha+0=\alpha,\\ \implies\ & \mathrm{d}(h\circ\iota_p)=\iota_p^*\mathrm{d}h=\iota_p^*\pi_1^*\alpha+\iota_p^*\pi_2^*\beta=(\pi_1\circ\iota_p)^*\alpha+(\pi_2\circ\iota_p)^*\beta=0+\mathrm{id}_N^*\beta=\beta.\end{aligned}$$

Thus, $[\alpha]=0\in H^1_{deR}(M)$, $[\beta]=0\in H^1_{deR}(N)$, and so Φ is injective.

Note 1: If M and N are connected, the homomorphism Φ is also surjective and thus an isomorphism. Let $p\in M$ and $q\in M$ be any points. For all $x\in M$ and $y\in N$, choose smooth paths

$$\gamma_x\colon[a_x,b_x]\longrightarrow M,\quad \gamma_y\colon[c_y,d_y]\longrightarrow N\quad \text{s.t.}\quad \gamma_x(a_x)=p,\quad \gamma_x(b_x)=x,\quad \gamma_y(c_y)=q,\quad \gamma_y(d_y)=y.$$

Suppose $\omega\in E^1(M\times N)$, $\mathrm{d}\omega=0$. By Stokes Theorem I (for singular simplicies applied twice),

$$\begin{aligned}0=\int_{\gamma_x\times\gamma_y}\mathrm{d}\big(\omega-\pi_1^*\iota_q^*\omega-\pi_2^*\iota_p^*\omega\big)&=\int_{\partial(\gamma_x\times\gamma_y)}\big(\omega-\pi_1^*\iota_q^*\omega-\pi_2^*\iota_p^*\omega\big)\\ &=\int_{\iota_q\circ\gamma_x+\iota_x\circ\gamma_y-\iota_y\circ\gamma_x-\iota_p\circ\gamma_y}\big(\omega-\pi_1^*\iota_q^*\omega-\pi_2^*\iota_p^*\omega\big).\end{aligned}$$

We define a function $h\colon M\times N\longrightarrow\mathbb{R}$ by

$$h(x,y)\equiv\int_{\gamma_y}\big(\iota_x^*\omega-\iota_x^*\pi_2^*\iota_p^*\omega\big)=\int_{\gamma_x}\big(\iota_y^*\omega-\iota_y^*\pi_1^*\iota_q^*\omega\big).$$

The equality above follows from the preceding displayed expression. By the first expression for $h(x,y)$, it does not depend on the choice of γ_x; by the second, it does not depend on the choice of γ_x. Thus, $h(x,y)$ is independent of the choices of γ_x and γ_y and thus defines a smooth function on $M\times N$ (because ω is a smooth form and γ_x and γ_y can be varied smoothly with x and y locally). By the independence of h of γ_y and the first expression for h above,

$$\iota_x^*\mathrm{d}h=\mathrm{d}(\iota_x^*h)=\iota_x^*\omega-\iota_x^*\pi_2^*\iota_p^*\omega=\iota_x^*\big(\omega-\pi_1^*\iota_q^*\omega-\pi_2^*\iota_p^*\omega\big)\qquad\forall\,x\in M.$$

By the independence of h of γ_x and the second expression for h above,

$$\iota_y^*\mathrm{d}h = \mathrm{d}(\iota_y^* h) = \iota_y^*\omega - \iota_y^*\pi_1^*\iota_q^*\omega = \iota_y^*\big(\omega - \pi_1^*\iota_q^*\omega - \pi_2^*\iota_p^*\omega\big) \qquad \forall\, y\in N.$$

Since the homomorphism

$$\iota_y^*\oplus\iota_x^*\colon T^*_{(x,y)}(M\times N)\longrightarrow T_x^*M\oplus T_y^*N$$

is an isomorphism for all $(x,y)\in M\times N$, it follows that

$$\mathrm{d}h = \omega - \pi_1^*\iota_q^*\omega - \pi_2^*\iota_p^*\omega \qquad\Longrightarrow\qquad [\omega] = \Phi\big([\iota_q^*\omega, \iota_p^*\omega]\big)\in H^1_{deR}(M\times N).$$

Thus, the homomorphism Φ is indeed surjective.

Note 2: The solution to the original question readily extends to show that the homomorphism

$$\Phi_k\colon H^k_{deR}(M)\oplus H^k_{deR}(N)\longrightarrow H^k_{deR}(M\times N), \qquad \big([\alpha],[\beta]\big)\longrightarrow \big[\pi_1^*\alpha+\pi_2^*\beta\big],$$

is well-defined and injective for all $k\ge 1$. However, Φ_k need not be surjective for $k>1$ even if M and N are connected. In fact, by the Kunneth formula, the homomorphism

$$\bigoplus_{\substack{i+j=k\\ i,j\ge 0}} H^i_{deR}(M)\times H^j_{deR}(N)\longrightarrow H^k_{deR}(M\times N), \qquad [\alpha]\otimes[\beta]\longrightarrow \big[\pi_1^*\alpha\wedge\pi_2^*\beta\big],$$

is an isomorphism if M and N are compact; this will be proved in Chapter 6 using Hodge Theory.

Problem 5 (25pts)

Let $V,W\longrightarrow M$ be smooth vector bundles over a smooth manifold M.

(a) *Suppose V is orientable. Show that W is orientable if and only if $V\oplus W$ is.*

(b) *Give an example of $V,W\longrightarrow M$ non-orientable so that $V\oplus W$ is orientable.*

(c) *Give an example of $V,W\longrightarrow M$ non-orientable so that $V\oplus W$ is non-orientable.*

For (b) and (c), specify M, V, and W and justify your answer; M need not be the same.

(a) A vector bundle is orientable if and only if its top exterior power is a trivial line bundle. Since V is orientable,

$$\Lambda^{\mathrm{top}}(V\oplus W)\approx\Lambda^{\mathrm{top}}V\otimes\Lambda^{\mathrm{top}}W\approx\tau_1\otimes\Lambda^{\mathrm{top}}W\approx\Lambda^{\mathrm{top}}W.$$

Thus, the line bundle $\Lambda^{\mathrm{top}}(V\oplus W)$ is trivial (and the vector bundle $V\oplus W$ is orientable) if and only if the line bundle $\Lambda^{\mathrm{top}}W$ is trivial (and the vector bundle W is orientable).

For (b) and (c), recall that the tautological line bundle $\gamma_1\longrightarrow\mathbb{R}P^1$ is not orientable; this is also the Mobius Band line bundle.

(b) If $V \longrightarrow M$ is any vector bundle, the vector bundle $V \oplus V \longrightarrow M$ is orientable because

$$\Lambda^{\mathrm{top}}(V \oplus V) \approx \Lambda^{\mathrm{top}} V \otimes \Lambda^{\mathrm{top}} V = (\Lambda^{\mathrm{top}} V)^{\otimes 2} \approx \tau_1 ,$$

since the tensor product of any real line bundle with itself is trivial. Since $\Lambda^{\mathrm{top}}(V \oplus V)$ is trivial, $V \oplus V$ is orientable. Another way to see this is to note that the vector bundle $V \oplus V \longrightarrow M$ admits a complex structure:

$$\mathfrak{i} : V \oplus V \longrightarrow V \oplus V, \qquad (v, w) \longrightarrow (-w, v).$$

Since $(V \oplus V, \mathfrak{i})$ is a complex vector bundle, $V \oplus V$ is orientable as a vector bundle.

Thus, as an example, we can take $V = W = \gamma_1 \longrightarrow \mathbb{R}P^1$.

(c) If $V \longrightarrow M$ and $W \longrightarrow N$ are non-orientable vector bundles, then neither are the vector bundles

$$\pi_1^* V, \pi_2^* W, V \times W = \pi_1^* V \oplus \pi_2^* W \longrightarrow M \times N,$$

where $\pi_1, \pi_2 : M \times N \longrightarrow M, N$ are the two projection maps. Here is why. Pick $p \in M$ and $q \in N$ and let

$$\iota_1 : M \longrightarrow M \times N, \quad x \longrightarrow (x, q), \qquad \iota_2 : N \longrightarrow M \times N, \quad y \longrightarrow (p, y),$$

be inclusions as horizontal and vertical slices. Since $\pi_1 \circ \iota_1 = \mathrm{id}_M$,

$$\iota_1^* \pi_1^* V = \{\pi_1 \circ \iota_1\}^* V = \mathrm{id}_M^* V = V \longrightarrow M$$

is a non-orientable vector bundle; thus, $\pi_1^* V \longrightarrow M \times N$ is also a non-orientable vector bundle. Similarly, $\iota_2^* \pi_2^* W = W \longrightarrow N$, and $\pi_2^* W \longrightarrow M \times N$ is a non-orientable vector bundle as well. On the other hand, $\pi_2 \circ \iota_1 : M \longrightarrow N$ is the constant map sending M to $q \in N$ and so

$$\iota_1^* \pi_2^* W = \{\pi_2 \circ \iota_1\}^* W \approx M \times W_q \longrightarrow M$$

is a trivial vector bundle and thus orientable. By part (a), the vector bundle

$$\iota_1^*(\pi_1^* V \oplus \pi_2^* W) = \iota_1^* \pi_1^* V \oplus \iota_1^* \pi_2^* W \longrightarrow M$$

is not orientable (because $\iota_1^* \pi_1^* V$ is not orientable, while $\iota_1^* \pi_2^* W$ is orientable). Therefore,

$$\pi_1^* V \oplus \pi_2^* W \longrightarrow M \times N$$

is a non-orientable vector bundle.

Thus, as an example, we can take

$$V = \pi_1^* \gamma_1, W = \pi_2^* \gamma_1 \longrightarrow \mathbb{R}P^1 \times \mathbb{R}P^1,$$

where $\pi_1, \pi_2 : \mathbb{R}P^1 \times \mathbb{R}P^1 \longrightarrow \mathbb{R}P^1$ are the projection maps.