MAT 531: Topology&Geometry, II
Spring 2011

Solutions to Problem Set 9

Problem 1 (20pts)

We have defined Čech cohomology for sheaves or presheaves of K-modules. All such objects are abelian. The sets $\check{H}^0$ and $\check{H}^1$ can be defined for sheaves or presheaves of non-abelian groups as well. The main example of interest is the sheaf $\mathcal{S}$ of germs of smooth (or continuous) functions to a Lie group G over a smooth manifold (or topological space) M.[1]

Let $\underline{U}=\{U_\alpha\}_{\alpha\in\mathcal{A}}$ be an open cover of M. Analogously to the abelian case, the set $\check{C}^k(\underline{U};\mathcal{S})$ of Čech k-cocycles is a group under pointwise multiplication of sections:

$$\cdot : \check{C}^k(\underline{U};\mathcal{S}) \times \check{C}^k(\underline{U};\mathcal{S}) \longrightarrow \check{C}^k(\underline{U};\mathcal{S}),$$
$$\{f\cdot g\}_{\alpha_0\alpha_1\ldots\alpha_k}(p) = f_{\alpha_0\alpha_1\ldots\alpha_k}(p)\cdot g_{\alpha_0\alpha_1\ldots\alpha_k}(p) \quad \forall\,\alpha_0,\alpha_1,\ldots,\alpha_k\in\mathcal{A},\ p\in U_{\alpha_0}\cap U_{\alpha_1}\cap\ldots\cap U_{\alpha_k}\,,$$

where $f_{\alpha_0\alpha_1\ldots\alpha_k}, g_{\alpha_0\alpha_1\ldots\alpha_k} : U_{\alpha_0}\cap U_{\alpha_1}\cap\ldots\cap U_{\alpha_k} \longrightarrow G$ are smooth (or continuous) functions (or equivalently sections of $\mathcal{S}$). The identity element $\mathbf{e}\in\check{C}^k(\underline{U};\mathcal{S})$ is given by

$$\mathbf{e}_{\alpha_0\alpha_1\ldots\alpha_k}(p) = \mathrm{id}_G \qquad \forall\,\alpha_0,\alpha_1,\ldots,\alpha_k\in\mathcal{A},\ p\in U_{\alpha_0}\cap U_{\alpha_1}\cap\ldots\cap U_{\alpha_k}\,.$$

Define the two bottom boundary maps by

$$\mathrm{d}_0\colon \check{C}^0(\underline{U};\mathcal{S}) \longrightarrow \check{C}^1(\underline{U};\mathcal{S}),\ (\mathrm{d}_0 f)_{\alpha_0\alpha_1} = f_{\alpha_0}\big|_{U_{\alpha_0}\cap U_{\alpha_1}}\cdot f_{\alpha_1}^{-1}\big|_{U_{\alpha_0}\cap U_{\alpha_1}}$$
$$\mathrm{d}_1\colon \check{C}^1(\underline{U};\mathcal{S}) \longrightarrow \check{C}^2(\underline{U};\mathcal{S}),\ (\mathrm{d}_1 g)_{\alpha_0\alpha_1\alpha_2} = g_{\alpha_1\alpha_2}\big|_{U_{\alpha_0}\cap U_{\alpha_1}\cap U_{\alpha_2}}\cdot g_{\alpha_0\alpha_2}^{-1}\big|_{U_{\alpha_0}\cap U_{\alpha_1}\cap U_{\alpha_2}}\cdot g_{\alpha_0\alpha_1}\big|_{U_{\alpha_0}\cap U_{\alpha_1}\cap U_{\alpha_2}},$$

for all $\alpha_0,\alpha_1,\alpha_2\in\mathcal{A}$. We also define an action of $\check{C}^0(\underline{U};\mathcal{S})$ on $\check{C}^1(\underline{U};\mathcal{S})$ by

$$*\colon \check{C}^0(\underline{U};\mathcal{S})\times\check{C}^1(\underline{U};\mathcal{S}) \longrightarrow \check{C}^1(\underline{U};\mathcal{S}),\ \ \{f*g\}_{\alpha_0\alpha_1} = f_{\alpha_0}\big|_{U_{\alpha_0}\cap U_{\alpha_1}}\cdot g_{\alpha_0\alpha_1}\cdot f_{\alpha_1}^{-1}\big|_{U_{\alpha_0}\cap U_{\alpha_1}} \in \Gamma(U_{\alpha_0}\cap U_{\alpha_1};\mathcal{S}).$$

Show that

(a) $\check{H}^0(\underline{U};\mathcal{S}) \equiv \ker\mathrm{d}_0 \equiv \mathrm{d}_0^{-1}(\mathbf{e})$ *is a subgroup of* $\check{C}^0(\underline{U};\mathcal{S})$;

(b) *for every Čech 1-cocycle g (i.e. $g\in\ker\mathrm{d}_1$) for an open cover $\underline{U}=\{U_\alpha\}_{\alpha\in\mathcal{A}}$,*

$$g_{\alpha\alpha} = \mathbf{e}|_{U_\alpha}, \quad g_{\alpha\beta}g_{\beta\alpha} = \mathbf{e}|_{U_\alpha\cap U_\beta}, \quad g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = \mathbf{e}|_{U_\alpha\cap U_\beta\cap U_\gamma}, \qquad \forall\,\alpha,\beta,\gamma\in\mathcal{A};$$

(c) $*$ *is a left action of $\check{C}^0(\underline{U};\mathcal{S})$ on $\check{C}^1(\underline{U};\mathcal{S})$ that restricts to an action on $\ker\mathrm{d}_1$ and*

$$\mathrm{Im}\,\mathrm{d}_0 \subset \check{C}^0(\underline{U};\mathcal{S})\mathbf{e}.$$

---

[1] A Lie group G is a smooth manifold and a group so that the group operations are smooth. Examples include $O(k)$, $SO(k)$, $U(k)$, $SU(k)$.

By part (c), we can define

$$\check{H}^1(\underline{U};\mathcal{S}) = \ker \mathrm{d}_1/\check{C}^0(\underline{U};\mathcal{S})\,;$$

this is a pointed set (a set with a distinguished element).

If $\underline{U}'=\{U'_\alpha\}_{\alpha\in\mathcal{A}'}$ *is a refinement of* $\underline{U}=\{U_\alpha\}_{\alpha\in\mathcal{A}}$*, any refining map* $\mu\colon \mathcal{A}'\longrightarrow\mathcal{A}$ *induces group homomorphisms*

$$\mu_k^*\colon \check{C}^k(\underline{U};\mathcal{S})\longrightarrow \check{C}^k(\underline{U}';\mathcal{S}),$$

which commute with d_0, d_1*, and the action of* $\check{C}^0(\cdot;\mathcal{S})$ *on* $\check{C}^1(\cdot;\mathcal{S})$*, similarly to Section 5.33. Thus,* μ *induces a group homomorphism and a map*

$$R^0_{\underline{U}',\underline{U}}\colon \check{H}^0(\underline{U};\mathcal{S})\longrightarrow \check{H}^0(\underline{U}';\mathcal{S}) \qquad\text{and}\qquad R^1_{\underline{U}',\underline{U}}\colon \check{H}^1(\underline{U};\mathcal{S})\longrightarrow \check{H}^1(\underline{U}';\mathcal{S}).$$

(d) *Show that these maps are independent of the choice of* μ.

Thus, we can again define $\check{H}^0(M;\mathcal{S})$ *and* $\check{H}^1(M;\mathcal{S})$ *by taking the direct limit of all* $\check{H}^0(\underline{U};\mathcal{S})$ *and* $\check{H}^1(\underline{U};\mathcal{S})$ *over open covers of* M*. The first set is a group, while the second need not be (unless* $\mathcal{S}$ *is a sheaf of abelian groups). These sets will be denoted by* $\check{H}^0(M;G)$ *and* $\check{H}^1(M;G)$ *if* $\mathcal{S}$ *is the sheaf of germs of smooth (or continuous) functions into a Lie group* G*. As in the abelian case,* $\check{H}^0(M;\mathcal{S})$ *is the space of global sections of* $\mathcal{S}$.

(e) *Show that there is a natural correspondence*

$$\big\{\text{isomorphism classes of rank } k \text{ real vector bundles over } M\big\}\longleftrightarrow \check{H}^1\big(M;O(k)\big).$$

(f) *What are the analogues of these statements for complex vector bundles? (state them and indicate the changes in the argument; do not re-write the entire solution).*

For $\alpha_0,\alpha_1,\ldots,\alpha_k\in\mathcal{A}$, let $U_{\alpha_0\alpha_1\ldots\alpha_k}=U_{\alpha_0}\cap U_{\alpha_1}\cap\ldots\cap U_{\alpha_k}$.

(a) If $f\in\ker\mathrm{d}_0$,

$$\begin{aligned}(\mathrm{d}_0f^{-1})_{\alpha_0\alpha_1}\equiv f^{-1}_{\alpha_0}\big|_{U_{\alpha_0\alpha_1}}\cdot f_{\alpha_1}\big|_{U_{\alpha_0\alpha_1}} &= \big(f^{-1}_{\alpha_1}\big|_{U_{\alpha_0\alpha_1}}\cdot f_{\alpha_0}\big|_{U_{\alpha_0\alpha_1}}\big)^{-1}\\ &= \big((\mathrm{d}_0f)_{\alpha_1\alpha_0}\big)^{-1}=\big(\mathbf{e}|_{U_{\alpha_1\alpha_0}}\big)^{-1}=\mathbf{e}|_{U_{\alpha_0\alpha_1}}\,;\end{aligned}$$

so $f^{-1}\in\ker\mathrm{d}_0$. If $f,\tilde{f}\in\ker\mathrm{d}_0$,

$$\begin{aligned}(\mathrm{d}_0(f\tilde{f}))_{\alpha_0\alpha_1}&\equiv (f\tilde{f})_{\alpha_0}\big|_{U_{\alpha_0\alpha_1}}(f\tilde{f})^{-1}_{\alpha_1}\big|_{U_{\alpha_0\alpha_1}} = f_{\alpha_0}\big|_{U_{\alpha_0\alpha_1}}\cdot\tilde{f}_{\alpha_0}\big|_{U_{\alpha_0\alpha_1}}\cdot\tilde{f}^{-1}_{\alpha_1}\big|_{U_{\alpha_0\alpha_1}}\cdot f^{-1}_{\alpha_1}\big|_{U_{\alpha_0\alpha_1}}\\ &= f_{\alpha_0}\big|_{U_{\alpha_0\alpha_1}}\cdot(\mathrm{d}_0\tilde{f})_{\alpha_0\alpha_1}\cdot f^{-1}_{\alpha_1}\big|_{U_{\alpha_0\alpha_1}} = f_{\alpha_0}\big|_{U_{\alpha_0\alpha_1}}\cdot\mathbf{e}|_{U_{\alpha_0\alpha_1}}\cdot f^{-1}_{\alpha_1}\big|_{U_{\alpha_0\alpha_1}}\\ &= (\mathrm{d}_0f)_{\alpha_0\alpha_1}=\mathbf{e}|_{U_{\alpha_0\alpha_1}};\end{aligned}$$

so $f\tilde{f}\in\ker\mathrm{d}_0$. Thus, $\ker\mathrm{d}_0\subset\check{C}^0(\underline{U};\mathcal{S})$ is a subgroup.

(b) If $g\in\ker\mathrm{d}_1$,

$$g_{\alpha_1\alpha_2}\big|_{U_{\alpha_0\alpha_1\alpha_2}}\cdot g^{-1}_{\alpha_0\alpha_2}\big|_{U_{\alpha_0\alpha_1\alpha_2}}\cdot g_{\alpha_0\alpha_1}\big|_{U_{\alpha_0\alpha_1\alpha_2}} = (\mathrm{d}_1g)_{\alpha_0\alpha_1\alpha_2}=\mathbf{e}|_{U_{\alpha_0\alpha_1\alpha_2}}\quad\forall\,\alpha_0,\alpha_1,\alpha_2.$$

Plugging in $(\alpha_0,\alpha_1,\alpha_2)=(\alpha,\alpha,\alpha)$, we obtain $g_{\alpha\alpha}=\mathbf{e}|_{U_\alpha}$. Plugging in $(\alpha_0,\alpha_1,\alpha_2)=(\beta,\alpha,\beta)$ and using $g_{\beta\beta}=\mathbf{e}|_{U_\beta}$, we obtain $g_{\alpha\beta}g_{\beta\alpha}=\mathbf{e}|_{U_{\alpha\beta}}$. Finally, plugging in $(\alpha_0,\alpha_1,\alpha_2)=(\gamma,\alpha,\beta)$ and using $g_{\beta\gamma}^{-1}=g_{\gamma\beta}$, we obtain

$$g_{\alpha\beta}|_{U_{\alpha\beta\gamma}}\cdot g_{\beta\gamma}|_{U_{\alpha\beta\gamma}}\cdot g_{\gamma\alpha}|_{U_{\alpha\beta\gamma}}=\mathbf{e}|_{U_{\alpha\beta\gamma}}.$$

(c) If $f,\tilde{f}\in\check{C}^0(\underline{U};\mathcal{S})$ and $g\in\check{C}^1(\underline{U};\mathcal{S})$, then

$$\begin{aligned}\big((f\tilde{f})*g\big)_{\alpha_0\alpha_1}&\equiv(f\tilde{f})_{\alpha_0}\big|_{U_{\alpha_0\alpha_1}}\cdot g_{\alpha_0\alpha_1}\cdot(f\tilde{f})^{-1}_{\alpha_1}\big|_{U_{\alpha_0\alpha_1}}=f_{\alpha_0}\big|_{U_{\alpha_0\alpha_1}}\cdot\tilde{f}_{\alpha_0}\big|_{U_{\alpha_0\alpha_1}}\cdot g_{\alpha_0\alpha_1}\cdot\tilde{f}_{\alpha_1}\big|_{U_{\alpha_0\alpha_1}}\cdot f_{\alpha_1}\big|_{U_{\alpha_0\alpha_1}}\\&=f_{\alpha_0}\big|_{U_{\alpha_0\alpha_1}}\cdot(\tilde{f}*g)_{\alpha_0\alpha_1}f_{\alpha_1}\big|_{U_{\alpha_0\alpha_1}}=\big(f*(\tilde{f}*g)\big)_{\alpha_0\alpha_1};\end{aligned}$$

so, $*$ is indeed a left action of $\check{C}^0(\underline{U};\mathcal{S})$ on $\check{C}^1(\underline{U};\mathcal{S})$. If $f\in\check{C}^0(\underline{U};\mathcal{S})$ and $g\in\ker\mathrm{d}_1$, then

$$\begin{aligned}\big(\mathrm{d}_1(f*g)\big)_{\alpha_0\alpha_1\alpha_2}&=(f*g)_{\alpha_1\alpha_2}\big|_{U_{\alpha_0\alpha_1\alpha_2}}\cdot(f*g)^{-1}_{\alpha_0\alpha_2}\big|_{U_{\alpha_0\alpha_1\alpha_2}}\cdot(f*g)_{\alpha_0\alpha_1}\big|_{U_{\alpha_0\alpha_1\alpha_2}}\\&=\big(f_{\alpha_1}\big|_{U_{\alpha_1\alpha_2}}g_{\alpha_1\alpha_2}f^{-1}_{\alpha_2}\big|_{U_{\alpha_1\alpha_2}}\big)\big|_{U_{\alpha_0\alpha_1\alpha_2}}\big(f_{\alpha_0}\big|_{U_{\alpha_0\alpha_2}}g_{\alpha_0\alpha_2}f^{-1}_{\alpha_2}\big|_{U_{\alpha_0\alpha_2}}\big)^{-1}\big|_{U_{\alpha_0\alpha_1\alpha_2}}\\&\qquad\times\big(f_{\alpha_0}\big|_{U_{\alpha_0\alpha_1}}g_{\alpha_0\alpha_1}f^{-1}_{\alpha_1}\big|_{U_{\alpha_0\alpha_1}}\big)\big|_{U_{\alpha_0\alpha_1\alpha_2}}\\&=f_{\alpha_1}\big|_{U_{\alpha_0\alpha_1\alpha_2}}g_{\alpha_1\alpha_2}\big|_{U_{\alpha_0\alpha_1\alpha_2}}g^{-1}_{\alpha_0\alpha_2}\big|_{U_{\alpha_0\alpha_1\alpha_2}}g_{\alpha_0\alpha_1}\big|_{U_{\alpha_0\alpha_1\alpha_2}}f^{-1}_{\alpha_1}\big|_{U_{\alpha_0\alpha_1\alpha_2}}\\&=f_{\alpha_1}\big|_{U_{\alpha_0\alpha_1\alpha_2}}\cdot(\mathrm{d}_1g)_{\alpha_0\alpha_1\alpha_2}\cdot f^{-1}_{\alpha_1}\big|_{U_{\alpha_0\alpha_1\alpha_2}}\\&=f_{\alpha_1}\big|_{U_{\alpha_0\alpha_1\alpha_2}}\cdot\mathbf{e}|_{U_{\alpha_0\alpha_1\alpha_2}}\cdot f^{-1}_{\alpha_1}\big|_{U_{\alpha_0\alpha_1\alpha_2}}=\mathbf{e}|_{U_{\alpha_0\alpha_1\alpha_2}}.\end{aligned}$$

Thus, $f*g\in\ker\mathrm{d}_1$ whenever $g\in\ker\mathrm{d}_1$. Since $\mathrm{d}_0f=f*\mathbf{e}$ for all $f\in\check{C}^0(\underline{U};\mathcal{S})$, $\mathrm{Im}\,\mathrm{d}_0\subset\check{C}^0(\underline{U};\mathcal{S})\mathbf{e}$.

(d) If $\mu\colon\mathcal{A}'\longrightarrow\mathcal{A}$ is a refining map, $U'_\alpha\subset U_{\mu(\alpha)}$ for every $\alpha\in\mathcal{A}'$. The group homomorphisms

$$\mu_k^*\colon\check{C}^k(\underline{U};\mathcal{S})\longrightarrow\check{C}^k(\underline{U}';\mathcal{S})$$

are defined by

$$\big\{\mu_0^*f\big\}_{\alpha_0\alpha_1\ldots\alpha_k}=f_{\mu(\alpha_0)\mu(\alpha_1)\ldots\mu(\alpha_k)}\big|_{U'_{\alpha_0\alpha_1\ldots\alpha_k}}\qquad\forall\,\alpha_0,\alpha_1,\ldots,\alpha_k\in\mathcal{A}'.$$

Suppose $\mu'\colon\mathcal{A}'\longrightarrow\mathcal{A}$ is another refining map, $U'_\alpha\subset U_{\mu(\alpha)\mu'(\alpha)}$ for every $\alpha\in\mathcal{A}'$. If $f\in\ker\mathrm{d}_0$, then

$$\begin{gathered}f_{\mu(\alpha_0)}\big|_{U_{\mu(\alpha_0)\mu'(\alpha_0)}}\cdot f^{-1}_{\mu'(\alpha_0)}\big|_{U_{\mu(\alpha_0)\mu'(\alpha_0)}}\equiv(\mathrm{d}_0f)_{\mu(\alpha_0)\mu'(\alpha_0)}=\mathbf{e}\big|_{U_{\mu(\alpha_0)\mu'(\alpha_0)}}\qquad\Longrightarrow\\f_{\mu(\alpha_0)}\big|_{U_{\mu(\alpha_0)\mu'(\alpha_0)}}=f_{\mu'(\alpha_0)}\big|_{U_{\mu(\alpha_0)\mu'(\alpha_0)}}\qquad\Longrightarrow\qquad(\mu_0^*f)_{\alpha_0}=f_{\mu(\alpha_0)}\big|_{U'_{\alpha_0}}=f_{\mu'(\alpha_0)}\big|_{U'_{\alpha_0}}=(\mu_0'^*f)_{\alpha_0}\\\Longrightarrow\qquad\mu_0^*=\mu_0'^*\colon\check{H}^0(\underline{U};\mathcal{S})=\ker\mathrm{d}_0\longrightarrow\check{H}^0(\underline{U}';\mathcal{S})\subset\check{C}^0(\underline{U}';\mathcal{S}).\end{gathered}$$

We next verify that $R^1_{\underline{U}',\underline{U}}$ is independent of μ. For each $g\in\check{C}^1(\underline{U};\mathcal{S})$, define

$$h_1g\in\check{C}^0(\underline{U}';\mathcal{S})\qquad\text{by}\qquad(h_1g)_\alpha=g_{\mu'(\alpha)\mu(\alpha)}|_{U'_\alpha}.$$

We will show that

$$\begin{aligned}
&\mu'^*g=(h_1g)*(\mu^*g) \quad \forall\, g\in\ker\mathrm{d}_1\\
\Longrightarrow \qquad &\mu_1^*=\mu_1'^*\colon \check{H}^1(\underline{U};\mathcal{S})=\ker\mathrm{d}_1/\check{C}^0(\underline{U};\mathcal{S}) \longrightarrow \check{H}^1(\underline{U'};\mathcal{S})=\ker\mathrm{d}_1/\check{C}^0(\underline{U'};\mathcal{S}).
\end{aligned}$$

If $g\in\ker\mathrm{d}_1\subset\check{C}^1(\underline{U};\mathcal{S})$, then

$$\begin{aligned}
g_{\mu(\alpha_0)\mu(\alpha_1)}\big|_{U_{\mu'(\alpha_1)\mu(\alpha_0)\mu(\alpha_1)}}\cdot g^{-1}_{\mu'(\alpha_1)\mu(\alpha_1)}\big|_{U_{\mu'(\alpha_1)\mu(\alpha_0)\mu(\alpha_1)}} &= g^{-1}_{\mu'(\alpha_1)\mu(\alpha_0)}\big|_{U_{\mu'(\alpha_1)\mu(\alpha_0)\mu(\alpha_1)}},\\
g_{\mu'(\alpha_0)\mu(\alpha_0)}\big|_{U_{\mu'(\alpha_1)\mu'(\alpha_0)\mu(\alpha_0)}}\cdot g^{-1}_{\mu'(\alpha_1)\mu(\alpha_0)}\big|_{U_{\mu'(\alpha_1)\mu'(\alpha_0)\mu(\alpha_0)}} &= g^{-1}_{\mu'(\alpha_1)\mu'(\alpha_0)}\big|_{U_{\mu'(\alpha_1)\mu'(\alpha_0)\mu(\alpha_0)}}\\
\Longrightarrow \quad \big((h_1g)*(\mu^*g)\big)_{\alpha_0\alpha_1} &= g_{\mu'(\alpha_0)\mu(\alpha_0)}|_{U'_{\alpha_0\alpha_1}}\cdot g_{\mu(\alpha_0)\mu(\alpha_1)}\big|_{U'_{\alpha_0\alpha_1}}\cdot g^{-1}_{\mu'(\alpha_1)\mu(\alpha_1)}|_{U'_{\alpha_0\alpha_1}}\\
&= g_{\mu'(\alpha_0)\mu(\alpha_0)}|_{U'_{\alpha_0\alpha_1}}\cdot g^{-1}_{\mu'(\alpha_1)\mu(\alpha_0)}\big|_{U'_{\alpha_0\alpha_1}}\\
&= g^{-1}_{\mu'(\alpha_1)\mu'(\alpha_0)}\big|_{U'_{\alpha_0\alpha_1}} = g_{\mu'(\alpha_0)\mu'(\alpha_1)}\big|_{U'_{\alpha_0\alpha_1}} = (\mu'^*g)_{\alpha_0\alpha_1}.
\end{aligned}$$

The second-to-last equality follows from part (b).

(e) Suppose $V\longrightarrow M$ is a real vector bundle of rank k. By Problem 5a on PS6, V admits a Riemannian metric; see also Section 11 in *Lecture Notes.* If

$$h_\alpha\colon V|_{U_\alpha}\longrightarrow U_\alpha\times\mathbb{R}^k$$

is a trivialization, by applying the Gramm-Schmidt procedure we can modify h_α so that it is metric-preserving with respect to the standard metric on $\mathbb{R}^k$. If $h_\beta\colon V|_{U_\beta}\longrightarrow U_\beta\times\mathbb{R}^k$ is another metric-preserving trivialization, the corresponding transition map

$$g_{\alpha\beta}\colon U_{\alpha\beta}\longrightarrow \mathrm{GL}_k\mathbb{R}$$

is an orthogonal transformation, i.e. $g_{\alpha\beta}\in C^\infty(U_{\alpha\beta};O(k))$.

Suppose $\underline{U}=\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is a cover of M such that $V|_{U_\alpha}$ is trivial for every $\alpha\in\mathcal{A}$. Choose an orthogonal trivialization h_α of $V|_{U_\alpha}$. The corresponding transition data

$$\big\{g_{\alpha\beta}\in C^\infty(U_{\alpha\beta};O(k))\colon \alpha,\beta\in\mathcal{A}\big\}$$

then determines an element $g\in\check{C}^1(\underline{U};O(k))$. By Section 9 in *Lecture Notes*,

$$g_{\alpha\alpha}=\mathbf{e}|_{U_\alpha},\quad g_{\alpha\beta}g_{\beta\alpha}=\mathbf{e}|_{U_{\alpha\beta}},\quad g_{\alpha\beta}|_{U_{\alpha\beta\gamma}}\cdot g_{\beta\gamma}|_{U_{\alpha\beta\gamma}}\cdot g_{\gamma\alpha}|_{U_{\alpha\beta\gamma}}=\mathbf{e}|_{U_{\alpha\beta\gamma}} \quad \forall\,\alpha,\beta,\gamma\in\mathcal{A}.$$

Therefore, $g\in\ker\mathrm{d}_1$ defines an element

$$[g_V]\in\check{H}^1(M;O(k)).$$

We will show that this element depends only on the isomorphism class of V.

If $\underline{U'}=\{U'_\alpha\}_{\alpha\in\mathcal{A}'}$ is a refinement of $\underline{U}=\{U_\alpha\}_{\alpha\in\mathcal{A}}$ and $\mu\colon\mathcal{A}'\longrightarrow\mathcal{A}$ is a refining map, then $h_{\mu(\alpha)}|_{U'_\alpha}$ is a trivialization of $V|_{U'_\alpha}$. The corresponding transition data is $g_{\mu(\alpha)\mu(\beta)}|_{U'_\alpha\cap U'_\beta}$, i.e. μ_1^*g. Since

$$[g]=[\mu_1^*g]\in\check{H}^1(M;O(k)),$$

it is sufficient to consider trivializations of isomorphic vector bundles over a common cover (otherwise we can simply take the intersections of open sets in the two covers). Suppose

$$\varphi\colon V \longrightarrow V'$$

is an isomorphism of vector bundles over M. It can be assumed that φ is an isometry (see the next paragraph). For each $\alpha\in\mathcal{A}$, let h'_α be a metric-preserving trivialization of $V'|_{U_\alpha}$. Denote by $g'\in\check{C}^1(\underline{U};O(k))$ the corresponding transition data. Then,

$$\tilde{f}_\alpha\equiv h'_\alpha\circ\varphi\circ h_\alpha^{-1}\colon U_\alpha\times\mathbb{R}^k\longrightarrow U_\alpha\times\mathbb{R}^k$$

is a diffeomorphism commuting with the projection map π_1 and restricting to an orthogonal transformation. Therefore,

$$\tilde{f}_\alpha(p,v)=\big(p,f_\alpha(p)\cdot v\big)\quad\forall(p,v)\in U_\alpha\times\mathbb{R}^k\qquad\text{for some}\qquad f_\alpha\in C^\infty(U_\alpha;O(k)),$$

i.e. $f\in\check{C}^0(\underline{U};O(k))$. We claim that

$$g'=f*g\qquad\Longrightarrow\qquad[g']=[g]\in\check{C}^1(\underline{U};O(k))\qquad\Longrightarrow\qquad[g_{V'}]=[g_V]\in\check{H}^1(M;O(k)).$$

For, if $\alpha,\beta\in\mathcal{A}$ and $(p,v)\in U_{\alpha\beta}$, by definition of $g_{\alpha\beta}$ and $g'_{\alpha\beta}$

$$\begin{aligned}\big(p,\{f*g\}_{\alpha\beta}(p)\cdot v\big)&=\big(p,\{f_\alpha(p)g_{\alpha\beta}(p)f_\beta^{-1}(p)\}\cdot v\big)=\tilde{f}_\alpha\big(p,\{g_{\alpha\beta}(p)f_\beta^{-1}(p)\}\cdot v\big)\\&=\big\{h'_\alpha\circ\varphi\circ h_\alpha^{-1}\big\}\big(\{h_\alpha\circ h_\beta^{-1}\}(p,f_\beta^{-1}(p)\cdot v)\big)=\big\{h'_\alpha\circ\varphi\circ h_\beta^{-1}\big\}\big(\tilde{f}_\beta^{-1}(p,v)\big)\\&=\big\{h'_\alpha\circ\varphi\circ h_\beta^{-1}\big\}\big(\{h_\beta\circ\varphi^{-1}\circ h'^{-1}_\beta\}(p,v)\big)\\&=\big\{h'_\alpha\circ h'^{-1}_\beta\big\}(p,v)=\big(p,g'_{\alpha\beta}(p)\cdot v\big).\end{aligned}$$

We now show that if $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$ are two metrics on a vector bundle V, there exists a vector bundle isomorphism $\varphi: V\longrightarrow V$ which is an isometry from the first metric to the second. This implies that isomorphic vector bundles endowed with Riemannian metrics are isometric as vector bundles. Let $\{(U_\alpha,h_\alpha)\}_{\alpha\in\mathcal{A}}$ be a system of trivializations which are orthogonal with respect to $\langle\cdot,\cdot\rangle$. For each $\alpha\in\mathcal{A}$, define

$$\begin{aligned}&P_\alpha\colon U_\alpha\longrightarrow \mathrm{GL}_k\mathbb{R}\qquad\text{by}\qquad v^t\cdot P_\alpha(p)\cdot w=\big\langle h_\alpha^{-1}(p,v),h_\alpha^{-1}(p,w)\big\rangle'\quad\forall\,p\in U_\alpha,\ v,w\in\mathbb{R}^k\\&\qquad\Longrightarrow\qquad P_\beta=g_{\alpha\beta}^t\cdot P_\alpha\cdot g_{\alpha\beta}=g_{\alpha\beta}^{-1}\cdot P_\alpha\cdot g_{\alpha\beta}\quad\forall\ \alpha,\beta\in\mathcal{A}.\end{aligned}$$

Since $P_\alpha(p)$ is a positive-definite symmetric matrix, it has a well-defined square root, i.e. a positive-definite symmetric matrix $f_\alpha(p)$ such that

$$P_\alpha(p)=f_\alpha(p)\cdot f_\alpha(p)=f_\alpha^t(p)\cdot f_\alpha(p).$$

Since P_α is smooth, so is f_α. By the above,

$$\begin{aligned}f_\beta^2=P_\beta&=g_{\alpha\beta}^t\cdot P_\alpha\cdot g_{\alpha\beta}=g_{\alpha\beta}^t\cdot f_\alpha\cdot f_\alpha\cdot g_{\alpha\beta}=g_{\alpha\beta}^tf_\alpha g_{\alpha\beta}\cdot g_{\alpha\beta}^tf_\alpha g_{\alpha\beta}\\&=\big(g_{\alpha\beta}^tf_\alpha g_{\alpha\beta}\big)^2.\end{aligned}$$

Since f_α is a positive-definite symmetric matrix, so is $g_{\alpha\beta}^t f_\alpha g_{\alpha\beta}$. Since f_β is also a positive-definite symmetric matrix, by the uniqueness of the square root

$$f_\beta = g_{\alpha\beta}^t f_\alpha g_{\alpha\beta} = g_{\alpha\beta}^{-1} f_\alpha g_{\alpha\beta} \qquad \Longrightarrow \qquad g_{\alpha\beta} = f_\alpha^{-1} g_{\alpha\beta} f_\beta \quad \forall\ \alpha,\beta\in\mathcal{A}.$$

We define a bundle map

$$\varphi\colon V \longrightarrow V \qquad \text{by} \qquad \varphi\big(h_\alpha^{-1}(p,v)\big) = h_\alpha^{-1}\big(p, f_\alpha^{-1}(p)\cdot v\big) \qquad \forall\ \alpha\in\mathcal{A},\ p\in U_\alpha,\ v\in\mathbb{R}^k.$$

This map is well-defined because

$$\begin{aligned} h_\alpha^{-1}(p,v) = h_\beta^{-1}(p,w) \qquad &\Longrightarrow \qquad v = g_{\alpha\beta}(p)\cdot w \qquad \Longrightarrow \\ h_\beta^{-1}\big(p, f_\beta^{-1}(p)\cdot w\big) = h_\beta^{-1}\big(p, g_{\alpha\beta}^{-1}(p) f_\alpha^{-1}(p) g_{\alpha\beta}(p)\cdot w\big) &= h_\beta^{-1}\big(p, g_{\alpha\beta}^{-1}(p)\cdot f_\alpha^{-1}(p)\cdot v\big) \\ &= h_\alpha^{-1}\big(p, f_\alpha^{-1}(p)\cdot v\big). \end{aligned}$$

It is an isomorphism of vector bundles because it restricts to isomorphisms of vector bundles on trivializations. Furthermore,

$$\begin{aligned} \big\langle \varphi(h_\alpha^{-1}(p,v)), \varphi(h_\alpha^{-1}(p,w))\big\rangle' &= \big\langle h_\alpha^{-1}(p, f_\alpha^{-1}(p)\cdot v), h_\alpha^{-1}(p, f_\alpha^{-1}(p)\cdot w)\big\rangle' \\ &= (f_\alpha^{-1}(p)v)^t P_\alpha(p)(f_\alpha^{-1}(p)w) = v^t\cdot f_\alpha^{-1}(p)^t f_\alpha(p)^t f_\alpha(p) f_\alpha^{-1}(p)\cdot w \\ &= v^t\cdot w = \big\langle h_\alpha^{-1}(p,v)), h_\alpha^{-1}(p,w)\big\rangle. \end{aligned}$$

The last equality holds because h_α is an isometry from $\langle\cdot,\cdot\rangle$ to the standard metric on $\mathbb{R}^k$. By the above equality, φ is an isometry from $(V,\langle\cdot,\cdot\rangle)$ to $(V,\langle\cdot,\cdot\rangle')$.

Conversely, given $[g]\in\check{H}^1(M;O(k))$, let $g\in\check{C}^1(\underline{U};O(k))$ be a representative for $[g]$. Since $g\in\ker\mathrm{d}_1$, by part (b) and Section 9 in *Lecture Notes*, g determines a vector bundle

$$V_g = \Big(\bigsqcup_{\alpha\in\mathcal{A}}\{\alpha\}\times U_\alpha\times\mathbb{R}^k\Big)\Big/\sim_g, \qquad \big(\alpha,p,g_{\alpha\beta}v\big)\sim_g\big(\beta,p,v\big) \quad \forall\ \alpha,\beta\in\mathcal{A},$$

with transition data g. We need to see that the isomorphism class $[V_g]$ of V_g depends only on $[g]$. First, if $\underline{U}'=\{U'_\alpha\}_{\alpha\in\mathcal{A}'}$ is a refinement of $\underline{U}=\{U_\alpha\}_{\alpha\in\mathcal{A}}$ and $\mu\colon\mathcal{A}'\longrightarrow\mathcal{A}$ is a refining map, then the vector bundles V_g and V_{μ^*g} as constructed in Section 9 are isomorphic. An isomorphism is given by

$$\varphi\colon V_{\mu^*g}=\Big(\bigsqcup_{\alpha\in\mathcal{A}'}\{\alpha\}\times U'_\alpha\times\mathbb{R}^k\Big)\Big/\sim_{\mu^*g}\longrightarrow V_g=\Big(\bigsqcup_{\alpha\in\mathcal{A}}\{\alpha\}\times U_\alpha\times\mathbb{R}^k\Big)\Big/\sim_g, \quad [\alpha,p,v]\longrightarrow\big[\mu(\alpha),p,v\big].$$

This map is well-defined because

$$(\alpha,p,v)\sim_{\mu^*g}(\beta,p,v') \qquad \Longrightarrow \qquad \big(\mu(\alpha),p,v\big)\sim_g\big(\mu(\beta),p,v'\big).$$

It is an isomorphism of vector bundles, since it is smooth, commutes with the projection maps, and its restriction to each fiber is an isomorphism. Thus, it is sufficient to show that if

$$g,g'\in\check{C}^1(\underline{U};O(k)) \qquad \text{and} \qquad [g]=[g']\in\check{H}^1(\underline{U};O(k)),$$

the vector bundles V_g and $V_{g'}$ are isomorphic. By definition, there exists

$$f\in\check{C}^0(\underline{U};O(k)) \qquad \text{s.t.} \qquad g'=f*g.$$

Define

$$\varphi\colon V_g=\Big(\bigsqcup_{\alpha\in\mathcal{A}}\{\alpha\}\times U'_\alpha\times\mathbb{R}^k\Big)\Big/\!\sim_g\longrightarrow V_{g'}=\Big(\bigsqcup_{\alpha\in\mathcal{A}}\{\alpha\}\times U_\alpha\times\mathbb{R}^k\Big)\Big/\!\sim_{g'}\qquad\text{by}$$

$$\varphi\big([\alpha,p,v]\big)=\big[\alpha,p,f_\alpha(p)\cdot v\big].$$

This map is well-defined, since

$$\begin{aligned}(\alpha,p,v)\sim_g(\beta,p,v') \quad&\Longrightarrow\quad v=g_{\alpha\beta}(p)\cdot v'\\ &\Longrightarrow\quad f_\alpha(p)\cdot v=f_\alpha(p)\cdot g_{\alpha\beta}(p)\cdot v'=\big\{(f*g)_{\alpha\beta}(p)\big\}\cdot f_\beta(p)\cdot v'\\ &\Longrightarrow\quad \big(\alpha,p,f_\alpha(p)\cdot v\big)\sim_{g'}\big(\beta,p,f_\beta(p)\cdot v'\big).\end{aligned}$$

Since φ is smooth, commutes with the projection maps, and its restriction to each fiber is an isomorphism, φ is an isomorphism of vector bundles.

It remains to observe that the two maps

$$\begin{aligned}\big\{\text{isomorphism classes of rank } k \text{ real vector bundles over } M\big\}\longrightarrow\check{H}^1\big(M;O(k)\big),\qquad&[V]\longrightarrow[g_V],\\ \check{H}^1\big(M;O(k)\big)\longrightarrow\big\{\text{isomorphism classes of rank } k \text{ real vector bundles over } M\big\},\qquad&[g]\longrightarrow[V_g],\end{aligned}$$

are mutual inverses. For, g is transition data for the vector bundle V_g and any vector bundle V is isomorphic to V_g if g is transition data for V; Section 9 in *Lecture Notes*.

(f) If V is a complex vector bundle, we can choose a Hermitian metric on V. If h_α and h_β are trivializations of V over U_α and U_β preserving the metric, the corresponding transition map $g_{\alpha\beta}\in C^\infty(U_{\alpha\beta};\mathrm{GL}_k\mathbb{C})$ is metric-preserving, i.e.

$$g_{\alpha\beta}\in C^\infty(U_{\alpha\beta};U(k)).$$

The rest of the argument in part (d) goes through, with $O(k)$ replaced by $U(k)$ and $\mathbb{R}$ by $\mathbb{C}$. So, we obtain a natural correspondence

$$\big\{\text{isomorphism classes of rank } k \text{ complex vector bundles over } M\big\}\longleftrightarrow\check{H}^1\big(M;U(k)\big),$$
$$[V]\longleftrightarrow[g_V],\qquad [V_g]\longleftrightarrow[g].$$

Problem 2 (15pts)

(a) *Show that the set of isomorphism classes of line bundles on M forms an abelian group under the tensor product (i.e. satisfies 3 properties for a group and another for abelian). Show that in the real case all nontrivial elements are of order two.*

(b) *Show that the correspondence*

$$\big\{\text{isomorphism classes of real line bundles over } M\big\}\longleftrightarrow\check{H}^1(M;\mathbb{Z}_2)$$

of the previous problem is a group isomorphism.

(c) *Show that there is a natural group isomorphism*

$$\big\{\text{isomorphism classes of complex line bundles over } M\big\} \longleftrightarrow \check{H}^2(M;\mathbb{Z}).$$

Note: The groups $\check{H}^1\big(M;\mathbb{Z}_2\big)$ and $\check{H}^2\big(M;\mathbb{Z}\big)$ are naturally isomorphic to the singular cohomology groups $H^1\big(M;\mathbb{Z}_2\big)$ and $H^2\big(M;\mathbb{Z}\big)$. The image of a real line bundle L

$$w_1(L) \in H^1(M;\mathbb{Z}_2)$$

is *the first Stiefel-Whitney class of* L; the image of a complex line bundle

$$c_1(L) \in H^2(M;\mathbb{Z})$$

is *the first Chern class of* L. However, this is not how these *characteristic classes* are normally defined.

(a) We need to show that the tensor product operation descends to isomorphism classes of line bundles, is associative and commutative, there is an identity element, and every element has an inverse. For the first property, we need to show that if L_1 is isomorphic to L_1' and L_2 is isomorphic to L_2', then $L_1\otimes L_2$ is isomorphic to $L_1'\otimes L_2'$. This is the case because if φ_1 is an isomorphism from L_1 to L_1' and φ_2 is an isomorphism from L_2 to L_2', then $\varphi_1\otimes\varphi_2$ is an isomorphism from $L_1\otimes L_2$ to $L_1'\otimes L_2'$. For the next two properties, define isomorphisms

$$\begin{array}{ccccc} L_1\otimes(L_2\otimes L_3) \longrightarrow (L_1\otimes L_2)\otimes L_3 & \text{and} & L_1\otimes L_2 \longrightarrow L_2\otimes L_1 & \text{by} \\ \big[v_1,[v_2,v_3]\big] \longrightarrow \big[[v_1,v_2],v_3\big] & \text{and} & [v_1,v_2] \longrightarrow [v_2,v_1]. & \end{array}$$

These bundle maps are smooth and isomorphisms on each fiber because they induce smooth maps on trivializations that are isomorphisms on every fiber. The identity element is represented by the trivial line bundle τ_1. If s is a nowhere-zero section of τ_1 (e.g. $s(p)=(p,1)$), the bundle map

$$L \longrightarrow L\otimes\tau_1, \qquad v \longrightarrow \big[v, s(\pi(v))\big],$$

is an isomorphism since it is injective. The inverse of $[L]$ is $[L^*]$, since the map

$$L^*\otimes L \longrightarrow \tau_1 = M\times\mathbb{R}, \qquad [v,\alpha] \longrightarrow \big(\pi(v), \alpha(v)\big),$$

is an isomorphism because it is surjective.

If L is a real line bundle and $\langle\cdot,\cdot\rangle$ is a Riemannian metric on L, the bundle map

$$\varphi\colon L \longrightarrow L^*, \qquad \big\{\varphi(v)\big\}(w) = \langle v, w\rangle,$$

is a vector bundle isomorphism because it induces smooth maps on trivializations that restrict to non-zero maps on every fiber. Since $L\otimes L^*$ is also isomorphic to τ_1, $L\otimes L$ is isomorphic to τ_1. This means that $[L]\otimes[L]$ is the identity element in the group of isomorphism classes of line bundles and therefore every nontrivial element is of order two.

Remark: This argument does not generalize to complex line bundles because the corresponding map φ would be $\mathbb{C}$-antilinear, instead of $\mathbb{C}$-linear. Complex line bundles are generally not of order two.

(b) By part (e) of the previous problem, there is a correspondence

$$\big\{\text{isomorphism classes of real line bundles over } M\big\} \longleftrightarrow \check{H}^1\big(M;O(1)\big)=\check{H}^1\big(M;\mathbb{Z}_2\big), \quad [L]\longleftrightarrow [g_L].$$

We need to show that this map is a group homomorphism, i.e.

$$[L]\otimes[L'] \longleftrightarrow [g_L]\cdot[g_{L'}];$$

here $\mathbb{Z}_2$ is viewed as the multiplicative group $\{\pm 1\}$. By definition, if $g=\{g_{\alpha\beta}\}$ and $g'=\{g'_{\alpha\beta}\}$ are transition data for L and L', then transition data for $L\otimes L'$ is given by

$$(g\otimes g')_{\alpha\beta} = g_{\alpha\beta}\otimes g'_{\alpha\beta} = g_{\alpha\beta}\cdot g'_{\alpha\beta}.$$

Thus,

$$[L]\otimes[L'] = [L\otimes L'] \longleftrightarrow [g\cdot g'] = [g]\cdot[g'],$$

i.e. this correspondence is a group isomorphism.

(c) Analogously to part (b), by part (f) of the previous problem there is a correspondence

$$\big\{\text{isomorphism classes of complex line bundles over } M\big\} \longleftrightarrow \check{H}^1\big(M;U(1)\big)=\check{H}^1\big(M;S^1\big),\ [L]\longleftrightarrow[g_L],$$

which is a group isomorphism. Recall from the statement of the previous problem that $\check{H}^1(M;S^1)$ is the first Čech cohomology for the sheaf $\mathfrak{C}^\infty(M;S^1)$ of germs of smooth functions to S^1. We have a short exact sequence of sheafs

$$0\longrightarrow \mathfrak{C}^\infty(M;\mathbb{Z})=M\times\mathbb{Z}\longrightarrow \mathfrak{C}^\infty(M;\mathbb{R})\longrightarrow \mathfrak{C}^\infty(M;S^1)\longrightarrow 0.$$

The first map is the inclusion of locally constant functions, while the second map is induced by the standard covering map

$$q\colon \mathbb{R}\longrightarrow S^1, \qquad q(t)=e^{2\pi i t}.$$

Thus, we obtain a long exact sequence in cohomology

$$\check{H}^1\big(M;\mathfrak{C}^\infty(M;\mathbb{R})\big)\longrightarrow \check{H}^1\big(M;\mathfrak{C}^\infty(M;S^1)\big)\xrightarrow{\delta}\check{H}^2\big(M;\mathfrak{C}^\infty(M;\mathbb{Z})\big)\longrightarrow \check{H}^2\big(M;\mathfrak{C}^\infty(M;\mathbb{R})\big).$$

Since $\mathfrak{C}^\infty(M;\mathbb{R})$ is a fine sheaf, the two outer groups vanish and therefore

$$\delta\colon \check{H}^1\big(M;S^1\big)=\check{H}^1\big(M;\mathfrak{C}^\infty(M;S^1)\big)\longrightarrow \check{H}^2\big(M;\mathfrak{C}^\infty(M;\mathbb{Z})\big)=\check{H}^2\big(M;\mathbb{Z}\big)$$

is an isomorphism. Combining with the above correspondence, we obtain a group isomorphism

$$\big\{\text{isomorphism classes of complex line bundles over } M\big\}\longrightarrow \check{H}^2\big(M;\mathbb{Z}\big), \qquad [L]\longrightarrow \delta\big([g_L]\big).$$

Problem 3: Chapter 2, #13 (10pts)

Let $(V, \langle\,\rangle)$ be an n-dimensional real inner-product space. Extend $\langle\,\rangle$ to all of ΛV by

$$\big\langle v_1\wedge\ldots v_k, w_1\wedge\ldots w_m\big\rangle = \begin{cases}\det(\langle v_i, w_j\rangle)_{i,j=1,\ldots,k}, & \textit{if } k=m;\\ 0, & \textit{otherwise.}\end{cases}$$

Since V is n-dimensional, $\Lambda^n V$ is one-dimensional. An orientation on *V is a choice of a component of $\Lambda^n V - 0$. Given such an orientation on V, a basis $\{e_1,\ldots,e_n\}$ for V is called* oriented *if $e_1\wedge\ldots\wedge e_n$ lies in the chosen component of $\Lambda^n V - \{0\}$. Define*

$$*\colon \Lambda V \longrightarrow \Lambda V$$

by requiring that for every oriented orthonormal basis $\{e_1,\ldots,e_n\}$ for V

$$*1 = e_1\wedge\ldots\wedge e_n, \qquad *\big(e_1\wedge\ldots\wedge e_n\big) = 1, \qquad *\big(e_1\wedge\ldots\wedge e_k\big) = e_{k+1}\wedge\ldots\wedge e_n.$$

Show that

(a) if $e_1,\ldots,e_n$ is an orthonormal basis for V, then

$$\big\{e_I\big\} \equiv \{1\}\cup\big\{e_{i_1}\wedge\ldots\wedge e_{i_k} : 1\le i_1<\ldots<i_k\le n\big\}$$

is an orthonormal basis for ΛV;

(b) $** = (-1)^{k(n-k)}$ *on $\Lambda^k V$;*

(c) $\langle v, w\rangle = *(v\wedge *w) = *(w\wedge *v)$ *for all $v, w\in V, W$.*

We can assume that the inner-product is positive-definite; otherwise, there is no orthonormal basis.

(a) First, all basis vectors are of unit length, since

$$\begin{aligned}\big\langle e_{i_1}\wedge\ldots e_{i_k}, e_{i_1}\wedge\ldots e_{i_k}\big\rangle &= \det(\langle e_{i_r}, e_{i_s}\rangle)_{r,s=1,\ldots,k}\\ &= \det(\delta_{i_r i_s})_{r,s=1,\ldots,k} = \det(\delta_{rs})_{r,s=1,\ldots,k} = \det \mathrm{I}_k = 1.\end{aligned}$$

Second, if two basis vectors are of different degree, their inner-product is zero by definition. On the other hand,

$$\big\langle e_{i_1}\wedge\ldots e_{i_k}, e_{j_1}\wedge\ldots e_{j_k}\big\rangle = \det(\langle e_{i_r}, e_{j_s}\rangle)_{r,s=1,\ldots,k} = \det(\delta_{i_r j_s})_{r,s=1,\ldots,k}.$$

If $(i_1,\ldots,i_k)\neq(j_1,\ldots,j_k)$, let r be the smallest number such that $i_r\neq j_r$. If $i_r<j_r$, then $\delta_{i_r j_s}=0$ for all s, since

$$j_s = i_s < i_r \;\text{ if }\; s<r \qquad\text{and}\qquad i_r < j_r \le j_s \;\text{ if }\; r\ge s.$$

Thus, all entries in the r-th row of the matrix $(\delta_{i_r j_s})_{r,s=1,\ldots,k}$ are zero. Similarly, if $i_r>j_r$, then all entries in the r-th column of the matrix $(\delta_{i_r j_s})_{r,s=1,\ldots,k}$ are zero. In either case,

$$\big\langle e_{i_1}\wedge\ldots e_{i_k}, e_{j_1}\wedge\ldots e_{j_k}\big\rangle = \det(\delta_{i_r j_s})_{r,s=1,\ldots,k} = 0.$$

Thus, the basis for ΛV is orthonormal.

We next show that the homomorphism $*$, as defined on the orthonormal basis vectors, exists (it then must be well-defined). We will show that it agrees with the values of a certain self-isomorphism of ΛV. Let μ be the unique unit vector in the chosen component of $\Lambda^n V$. Define bilinear map

$$A\colon \Lambda V\times\Lambda V \longrightarrow \mathbb{R} \quad \text{by} \quad A(v,w)=0 \;\text{ if }\; v\wedge w\notin \Lambda^n V, \quad v\wedge w = A(v,w)\mu \;\text{ if }\; v\wedge w\in\Lambda^n V.$$

This pairing is non-singular, since if $\{e_1,\ldots,e_n\}$ is an orthonormal basis for V, then

$$e_I\wedge(*e_J)=\pm\delta_{IJ}\mu \qquad \Longrightarrow \qquad A\big(e_I,*e_J\big)=\pm\delta_{IJ}.$$

Thus, by 2.7, A induces an isomorphism

$$T_A\colon \Lambda V\longrightarrow(\Lambda V)^*, \qquad (T_Aw)(v)=A(v,w) \;\; \forall\, v,w\in\Lambda V.$$

Since the inner-product on ΛV is nondegenerate, the pairing

$$B\colon \Lambda V\times\Lambda V\longrightarrow\mathbb{R}, \qquad B(v,w)=\langle v,w\rangle,$$

is non-singular as well and induces an isomorphism

$$T_B\colon \Lambda V\longrightarrow(\Lambda V)^*, \qquad (T_Bw)(v)=B(v,w) \;\; \forall\, v,w\in\Lambda V.$$

We claim that the isomorphism $T_A^{-1}\circ T_B\colon \Lambda V\longrightarrow\Lambda V$ satisfies

$$T_A^{-1}\circ T_B\big(e_I\big)=*e_I$$

for every oriented orthonormal basis $e_1,\ldots,e_n$. We need to show that

$$\begin{aligned} T_A(*e_I)=T_B(e_I)\in(\Lambda V)^* \qquad &\Longleftrightarrow \qquad T_A(*e_I)(e_J)=T_B(e_I)(e_J)\\ &\Longleftrightarrow \qquad A(e_J,*e_I)=\big\langle e_J,e_I\big\rangle. \end{aligned}$$

for all basis vectors e_I and e_J for ΛV. Suppose $I=(i_1,\ldots,i_k)=(1,\ldots,k)$, which we can assume after reordering the basis elements and possibly changing the sign of one of them. If $e_J\notin\Lambda^kV$, then

$$\big\langle e_J,e_I\big\rangle=0; \qquad e_J\wedge(*e_I)\notin\Lambda^n \quad\Longrightarrow\quad A(e_J,*e_I)=0.$$

On other hand, if $e_J=e_{j_1}\wedge\ldots\wedge e_{j_k}$, then

$$e_J\wedge(*e_I)=\delta_{(j_1,\ldots,j_k),(1,\ldots,k)}e_1\wedge\ldots\wedge e_n=\delta_{IJ}\mu \qquad\Longrightarrow\qquad A(e_J,*e_I)=\delta_{IJ},$$

while $\langle e_J,e_I\rangle=\delta_{IJ}$ by part (a). We have thus verified the claim.

(c) By the above, for all $v,w\in\Lambda V$,

$$\begin{aligned} *(v\wedge *w)=A(v,*w)=\big\{T_A(*w)\big\}(v)=(T_Bw)(v)=B(v,w)&=\langle v,w\rangle\\ &=\langle w,v\rangle=B(w,v)=*(w\wedge *v), \end{aligned}$$

by symmetry.

(b) Suppose $\{e_1,\ldots,e_n\}$ is an oriented orthonormal basis for V. Then,

$$\begin{aligned} &\mu=e_1\wedge\ldots\wedge e_n=\big(e_{k+1}\wedge\ldots\wedge e_n\big)\wedge\big((-1)^{k(n-k)}e_1\wedge\ldots\wedge e_k\big)\\ \Longrightarrow \qquad &**\,e_1\wedge\ldots\wedge e_k=*\big(e_{k+1}\wedge\ldots\wedge e_n\big)=(-1)^{k(n-k)}e_1\wedge\ldots\wedge e_k\\ \Longrightarrow \qquad &**=(-1)^{k(n-k)}\colon \Lambda^kV\longrightarrow\Lambda^kV. \end{aligned}$$