1. We have defined Čech cohomology for sheaves or presheaves of $K$-modules. All such objects are abelian. The sets $\check{H}^0$ and $\check{H}^1$ can be defined for sheaves or presheaves of non-abelian groups as well. The main example of interest is the sheaf $S$ of germs of smooth (or continuous) functions to a Lie group $G$ over a smooth manifold (or topological space) $M$.\footnote{A Lie group $G$ is a smooth manifold and a group so that the group operations are smooth. Examples include $O(k), SO(k), U(k), SU(k)$.}

Let $\mathcal{U} = \{ U_\alpha \}_{\alpha \in A}$ be an open cover of $M$. Analogously to the abelian case, the set $\check{C}^k(\mathcal{U}; S)$ of Čech $k$-cocycles is a group under pointwise multiplication of sections:

\[
\cdots: \check{C}^k(\mathcal{U}; S) \times \check{C}^k(\mathcal{U}; S) \to \check{C}^k(\mathcal{U}; S),
\]

\[
\{ f \cdot g \}_{\alpha_0 \ldots \alpha_k}(p) = f_{\alpha_0 \ldots \alpha_k}(p) \cdot g_{\alpha_0 \ldots \alpha_k}(p) \quad \forall \alpha_0, \alpha_1, \ldots, \alpha_k \in A, \ p \in U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_k},
\]

where $f_{\alpha_0 \ldots \alpha_k}, g_{\alpha_0 \ldots \alpha_k}: U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_k} \to G$ are smooth (or continuous) functions (or equivalently sections of $S$). The identity element $e \in \check{C}^k(\mathcal{U}; S)$ is given by

\[
e_{\alpha_0 \ldots \alpha_k}(p) = id_G \quad \forall \alpha_0, \alpha_1, \ldots, \alpha_k \in A, \ p \in U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_k}.
\]

Define the two bottom boundary maps by

\[
d_0: \check{C}^0(\mathcal{U}; S) \to \check{C}^1(\mathcal{U}; S), \quad (d_0 f)_{\alpha_0 \alpha_1} = f_{\alpha_0 | U_{\alpha_0} \cap U_{\alpha_1}} \cdot f_{\alpha_1}^{-1} | U_{\alpha_0} \cap U_{\alpha_1}
\]

\[
d_1: \check{C}^1(\mathcal{U}; S) \to \check{C}^2(\mathcal{U}; S), \quad (d_1 g)_{\alpha_0 \alpha_1 \alpha_2} = g_{\alpha_1 \alpha_2 | U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_2}} \cdot g_{\alpha_0 \alpha_2}^{-1} | U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_2} \cdot g_{\alpha_0 \alpha_1} | U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_2}
\]

for all $\alpha_0, \alpha_1, \alpha_2 \in A$. We also define an action of $\check{C}^0(\mathcal{U}; S)$ on $\check{C}^1(\mathcal{U}; S)$ by

\[
*: \check{C}^0(\mathcal{U}; S) \times \check{C}^1(\mathcal{U}; S) \to \check{C}^1(\mathcal{U}; S), \quad \{ fg \}_{\alpha_0 \alpha_1} = f_{\alpha_0 | U_{\alpha_0} \cap U_{\alpha_1}} \cdot g_{\alpha_0 \alpha_1} \cdot f_{\alpha_1}^{-1} | U_{\alpha_0} \cap U_{\alpha_1} \in \Gamma(U_{\alpha_0} \cap U_{\alpha_1}; S).
\]

Show that

(a) $\check{H}^0(\mathcal{U}; S) \equiv \ker d_0 \equiv d_0^{-1}(e)$ is a subgroup of $\check{C}^0(\mathcal{U}; S)$;

(b) for every Čech 1-cocycle $g$ (i.e. $g \in \ker d_1$) for an open cover $\mathcal{U} = \{ U_\alpha \}_{\alpha \in A}$,

\[
g_{\alpha \alpha} = e | U_\alpha, \quad g_{\alpha \beta} g_{\beta \alpha} = e | U_\alpha \cap U_\beta, \quad g_{\alpha \beta} g_{\beta \gamma} g_{\gamma \alpha} = e | U_\alpha \cap U_\beta \cap U_\gamma, \quad \forall \alpha, \beta, \gamma \in A;
\]

(c) $*$ is a left action of $\check{C}^0(\mathcal{U}; S)$ on $\check{C}^1(\mathcal{U}; S)$ that restricts to an action on $\ker d_1$ and

\[
\Im d_0 \subset \check{C}^0(\mathcal{U}; S)e.
\]

By part (c), we can define

\[
\check{H}^1(\mathcal{U}; S) = \ker d_1 / \check{C}^0(\mathcal{U}; S);
\]

this is a pointed set (a set with a distinguished element).
If $\mathcal{U}' = \{U'_\alpha\}_{\alpha \in \mathcal{A}'}$ is a refinement of $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$, any refining map $\mu : \mathcal{A}' \to \mathcal{A}$ induces group homomorphisms

$$\mu^*_k : \check{C}^k(\mathcal{U}; S) \to \check{C}^k(\mathcal{U}'; S),$$

which commute with $d_0$, $d_1$, and the action of $\check{C}^0(\cdot; S)$ on $\check{C}^1(\cdot; S)$, similarly to Section 5.33. Thus, $\mu$ induces a group homomorphism and a map

$$R^0_{\mathcal{U}', \mathcal{U}} : \check{H}^0(\mathcal{U}; S) \to \check{H}^0(\mathcal{U}'; S) \quad \text{and} \quad R^1_{\mathcal{U}', \mathcal{U}} : \check{H}^1(\mathcal{U}; S) \to \check{H}^1(\mathcal{U}'; S).$$

(d) Show that these maps are independent of the choice of $\mu$.

Thus, we can again define $\check{H}^0(M; S)$ and $\check{H}^1(M; S)$ by taking the direct limit of all $\check{H}^0(\mathcal{U}; S)$ and $\check{H}^1(\mathcal{U}; S)$ over open covers of $M$. The first set is a group, while the second need not be (unless $S$ is a sheaf of abelian groups). These sets will be denoted by $\check{H}^0(M; G)$ and $\check{H}^1(M; G)$ if $S$ is the sheaf of germs of smooth (or continuous) functions into a Lie group $G$. As in the abelian case, $\check{H}^0(M; S)$ is the space of global sections of $S$.

(e) Show that there is a natural correspondence

$$\{\text{isomorphism classes of rank } k \text{ real vector bundles over } M\} \leftrightarrow \check{H}^1(M; O(k)).$$

(f) What are the analogues of these statements for complex vector bundles? (state them and indicate the changes in the argument; do not re-write the entire solution).

**Hint:** For (e) and (f), you might want to look over Sections 9 and 11 in Lecture Notes. Do not forget that $\check{H}^1(M; S)$ is a direct limit.

2. (a) Show that the set of isomorphism classes of line bundles on $M$ forms an abelian group under the tensor product (i.e. satisfies 3 properties for a group and another for abelian). Show that in the real case all nontrivial elements are of order two.

(b) Show that the correspondence

$$\{\text{isomorphism classes of real line bundles over } M\} \leftrightarrow \check{H}^1(M; \mathbb{Z}_2)$$

of the previous problem is a group isomorphism.

(c) Show that there is a natural group isomorphism

$$\{\text{isomorphism classes of complex line bundles over } M\} \leftrightarrow \check{H}^2(M; \mathbb{Z}).$$

**Hint:** Snake Lemma.

**Note:** The groups $\check{H}^1(M; \mathbb{Z}_2)$ and $\check{H}^2(M; \mathbb{Z})$ are naturally isomorphic to the singular cohomology groups $H^1(M; \mathbb{Z}_2)$ and $H^2(M; \mathbb{Z})$. The image of a real line bundle $L$, $w_1(L) \in H^1(M; \mathbb{Z}_2)$, is the first Stiefel-Whitney class of $L$; the image of a complex line bundle, $c_1(L) \in H^2(M; \mathbb{Z})$, is the first Chern class of $L$. However, this is not how these characteristic classes are normally defined.

3. Chapter 2, #13 (p79)

**Exercise** (figure this out, but do not hand it in): Chapter 5, #20 (p217).