Problem Set 8
Due on Thursday, 4/14, in class

1. Suppose $X$ is a topological space and $\mathcal{P} = \{S_U; \rho_{U,V}\}$ is a presheaf on $X$. Let

$$\tilde{S}_U = \{(U_\alpha, f_\alpha)_{\alpha \in \mathcal{A}} : U_\alpha \subset U \text{ open}, U = \bigcup_{\alpha \in \mathcal{A}} U_\alpha; f_\alpha \in S_{U_\alpha};$$

$$\forall \alpha, \beta \in \mathcal{A}, p \in U_\alpha \cap U_\beta \exists W \subset U_\alpha \cap U_\beta \text{ open s.t. } p \in W, \rho_{W,U_\alpha} f_\alpha = \rho_{W,U_\beta} f_\beta\} / \sim,$$

where $(U_\alpha, f_\alpha)_{\alpha \in \mathcal{A}} \sim (U'_\alpha, f'_\alpha)_{\alpha' \in \mathcal{A}'}$ if $\forall \alpha \in \mathcal{A}, \alpha' \in \mathcal{A}', p \in U_\alpha \cap U'_\alpha'$

$$\exists W \subset U_\alpha \cap U'_\alpha' \text{ s.t. } p \in W, \rho_{W,U_\alpha} f_\alpha = \rho_{W,U'_\alpha'} f'_\alpha'.$$

Whenever $U \subset V$ are open subsets of $X$, the homomorphisms $\rho_{U,V}$ induce homomorphisms

$$\tilde{\rho}_{U,V} : \tilde{S}_V \to \tilde{S}_U, \quad [(V_\alpha, f_\alpha)_{\alpha \in \mathcal{A}}] \to [(V_\alpha \cap U, \rho_{V_\alpha \cap U,V_\alpha} f_\alpha)_{\alpha \in \mathcal{A}}],$$

so that $\tilde{\mathcal{P}} \equiv \{\tilde{S}_X; \tilde{\rho}_{U,V}\}$ is a presheaf on $X$. Show that

(a) $\tilde{\mathcal{P}} = \alpha(\beta(\mathcal{P}))$;

(b) the presheaf homomorphism $\{\varphi_U\} : \mathcal{P} \to \tilde{\mathcal{P}}$

$$\varphi_U : S_U \to \tilde{S}_U, \quad f \to [(U, f)],$$

is injective (resp. an isomorphism) if and only if $\mathcal{P}$ satisfies 5.7(C$_1$) (resp. is complete);

(c) if $\mathcal{R}$ is a subsheaf of $\mathcal{S}$, then $\alpha(S/\mathcal{R}) \approx \alpha(S)/\alpha(\mathcal{R})$.

Hint: see 5.8 for (b) and Chapter 5 #2,5 (p216) for (c).

2. Chapter 5, #17 (p217); hint: this is barely a two-liner, including justification.

3. Let $K$ be any ring containing 1. For each $i \in \mathbb{Z}^+$, let $V_i = K$; this is a $K$-module. Whenever $i \leq j$, define

$$\rho_{ji} : V_i \to V_j \quad \text{by} \quad \rho_{ji}(v) = 2^{j-i}v;$$

this is a homomorphism of $K$-modules. Since $\rho_{ki} = \rho_{kj} \rho_{ji}$ whenever $i \leq j \leq k$, we have a directed system and get a direct-limit $K$-module

$$V_\infty = \varinjlim_{\mathbb{Z}^+} V_i = \lim_{i \to \infty} V_i.$$

(a) Suppose $2 = 0 \in K$ (e.g. $K = \mathbb{Z}_2$). Show that $V_\infty = \{0\}$.

(b) Suppose $2$ is a unit in $K$ (e.g. $K = \mathbb{R}$). Show that $V_\infty \approx K$ as $K$-modules.

(c) Suppose 2 is not a unit in $K$, but $2 \neq 0 \in K$, and $K$ is an integral domain (e.g. $K = \mathbb{Z}$). Show that the $K$-module $V_\infty$ is not finitely generated.

Note: if you prefer, you can do the e.g. cases; this makes no difference in the argument.