1. Let $V$ be a vector space of dimension $n$ and $\Omega \in \Lambda^n V^*$ a nonzero element. Show that the homomorphism

$$V \rightarrow \Lambda^{n-1} V^*, \quad v \mapsto i_v \Omega,$$

where $i_v$ is the contraction map, is an isomorphism.

2. Suppose $M$ is a smooth $n$-manifold.

   (a) Let $\Omega$ be a nowhere-zero $n$-form on $M$. Show that for every $p \in M$ there exists a chart $(x_1, \ldots, x_n): U \rightarrow \mathbb{R}^n$ around $p$ such that

   $$\Omega|_U = dx_1 \wedge \ldots \wedge dx_n.$$

   (b) Let $\alpha$ be a nowhere-zero closed $(n-1)$-form on $M$. Show that for every $p \in M$ there exists a chart $(x_1, \ldots, x_n): U \rightarrow \mathbb{R}^n$ around $p$ such that

   $$\alpha|_U = dx_2 \wedge dx_3 \wedge \ldots \wedge dx_n.$$

3. Let $M$ be a smooth manifold and $X, Y \in \Gamma(M; TM)$ smooth vector fields on $M$. Show that the Lie derivative satisfies

$$L_{[X,Y]} = [L_X, L_Y] \equiv L_X \circ L_Y - L_Y \circ L_X$$

as homomorphisms on $\Gamma(M; TM)$ and $E^k(M)$. Hint: use 1.44,1.45d, 2.25abe.

4. Let $\alpha$ be a $k$-form on a smooth manifold $M$ and $X_0, X_1, \ldots, X_k$ smooth vector fields on $M$. Show directly from the definitions that

$$d\alpha(X_0, X_1, \ldots, X_k) = \sum_{i=0}^{i=k} (-1)^i X_i(\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_k))$$

$$+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k).$$

Hint: first show that the values of LHS and RHS at any $p \in M$ depend only on the values of $X_0, X_1, \ldots, X_k$ at $p$.

5. Let $V \rightarrow M$ be a smooth vector bundle of rank $k$ and $W \subset V$ a smooth subbundle of $V$ of rank $k'$. Show that

$$\text{Ann}(W) \equiv \{ \alpha \in V^*_p: \alpha(w) = 0 \forall w \in W, p \in M \}$$

is a smooth subbundle of $V^*$ of rank $k-k'$.
6. Suppose $M$ is a 3-manifold, $\alpha$ is a nowhere-zero one-form on $M$, and $p \in M$. Show that

(a) if there exists an embedded 2-dimensional submanifold $P \subset M$ such that $p \in P$ and $\alpha|_{TP} = 0$, then $(\alpha \wedge d\alpha)|_p = 0$.

(b) if there exists a neighborhood $U$ of $p$ in $M$ such that $(\alpha \wedge d\alpha)|_U = 0$, then there exists an embedded 2-dimensional submanifold $P \subset M$ such that $p \in P$ and $\alpha|_{TP} = 0$.

Note: If the top form $\alpha \wedge d\alpha$ on $M$ is nowhere-zero, $\alpha$ is called a contact form. In this case, it has no integrable submanifolds at all.

7. A two-form $\omega$ on a smooth manifold $M$ is called symplectic if $\omega$ is closed (i.e. $d\omega = 0$) and everywhere nondegenerate\(^1\). Suppose $\omega$ is a symplectic form on $M$.

(a) Show that the dimension of $M$ is even and the map

$$TM \rightarrow T^*M, \quad X \mapsto i_X \omega,$$

is a vector bundle isomorphism ($i_X$ is the contraction w.r.t. $X$, i.e. the dual of $X \wedge$).

(b) If $H : M \rightarrow \mathbb{R}$ is a smooth map, let $X_H \in \Gamma(M; TM)$ be the preimage of $dH$ under this isomorphism. Assume that $X_H$ is a complete vector field, so that the flow

$$\varphi : \mathbb{R} \times M \rightarrow M, \quad (t, p) \mapsto \varphi_t(p),$$

is globally defined. Show that for every $t \in \mathbb{R}$, the time-$t$ flow $\varphi_t : M \rightarrow M$ is a symplectomorphism, i.e. $\varphi_t^* \omega = \omega$.

Note: In such a situation, $H$ is called a Hamiltonian and $\varphi_t$ a Hamiltonian symplectomorphism.

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\(^1\)This means that $\omega_p \in \Lambda^2 T^*_p M$ is nondegenerate for every $p \in M$, i.e. for every $v \in T_p M = 0$ there exists $v' \in T_p M$ such that $\omega_p(v, v') \neq 0.$