Problem 1: Chapter 1, #13ad (10pts)

(a) Show that $[X,Y]$ is a smooth vector field on $M$ for any two smooth vector fields $X$ and $Y$ on $M$.

(d) Show that $[,]$ satisfies the Jacobi identity, i.e.

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$$

for all smooth vector fields on $X$, $Y$, and $Z$ on $M$.

(a) First, we need to see that for every $p \in M$ the map

$$[X,Y]_p: C^\infty(M) \to \mathbb{R}, \quad [X,Y]_p(f) = X_p(Yf) - Y_p(Xf),$$

is well-defined and is an element of $T_p M$, i.e. it is bilinear, satisfies the product rule, and its value depends only on the germ of $f$ at $p$. Since $X$, $Y$, and $f$ are smooth, $Yf$ and $Xf$ are smooth functions on $M$ by Proposition 1.43. Since $X_p$ and $Y_p$ are linear functionals on $C^\infty(M)$,

$$X_p(Yf), Y_p(Xf) \in \mathbb{R} \implies [X,Y]_p(f) \in \mathbb{R},$$

i.e. the map $[X,Y]_p$ is well-defined. Since it is a composition of linear maps, $[X,Y]_p$ is a linear map as well. Furthermore, if $f, g \in C^\infty(M)$,

$$[X,Y]_p(fg) = X_p(Y(fg)) - Y_p(X(fg)) = X_p(f Y(g) + g Y(f)) - Y_p(f X(g) + g X(f))$$

$$= (f(p)X_p(Y(g)) + Y_p(g)X_p(f) + g(p)X_p(Y(f)) + Y_p(f)X_p(g))$$

$$- (f(p)Y_p(X(g)) + X_p(g)Y_p(f) + g(p)Y_p(X(f)) + X_p(f)Y_p(g))$$

$$= f(p)(X_p(Y(g)) - Y_p(X(g))) + g(p)(X_p(Y(f)) - Y_p(X(f)))$$

$$= f(p)[X,Y]_p(g) + g(p)[X,Y]_p(f),$$

i.e. the linear map $[X,Y]_p$ satisfies the product rule. Finally, if $U$ is a neighborhood of $p$ in $M$ and $f|_U = g|_U$, then

$$(Xf)|_U = (Xg)|_U \quad \text{ and } \quad (Yf)|_U = (Yg)|_U,$$

since for all $q \in U$ the real numbers $X_qf$ and $X_qg$ depend only on the germs of $f$ and $g$ at $q$. Since the values of $X_p$ and $Y_p$ on $C^\infty(M)$ depend only on the germs of functions at $q$, we conclude that

$$Y_p(Xf) = Y_p(Xg) \quad \text{ and } \quad X_p(Yf) = X_p(Yg) \implies [X,Y]_p f = [X,Y]_p g,$$

i.e. the value of $[X,Y]_p$ on $f \in C^\infty(M)$ depends only on the germ of $f$ at $p$. Thus, $[X,Y]$ is a vector field on $M$.

If $X$, $Y$, and $f$ are smooth, then $Xf$ and $Yf$ are smooth functions on $M$ by Proposition 1.43. By Proposition 1.43 again, $Y(Xf)$ and $X(Yf)$ are also smooth functions on $M$. It follows that the function $[X,Y]f$ is smooth for every smooth function $f$ on $M$. Thus, $[X,Y]$ is a smooth vector
field on $M$ by Proposition 1.43.

(d) We need to show that LHS of the identity is the zero map, i.e. the function obtained by applying LHS to any smooth function $f$ on $M$ is zero. The first summand gives:

\[
[X,Y,Z]f = [X,Y](Zf) - Z([X,Y]f) - Z(X(Yf) - Y(Xf))
\]

\[
= X(Y(Zf)) - Y(X(Zf)) - Z(X(Yf)) + Z(Y(Xf)).
\]

Permuting $X$, $Y$, and $Z$ cyclicly, we then obtain

\[
[Y,Z,X]f = Y(Z(Xf)) - Z(Y(Xf)) - X(Y(Zf)) + X(Z(Yf))
\]

and

\[
[Z,X,Y]f = Z(X(Yf)) - X(Z(Yf)) - Y(Z(Xf)) + Y(X(Zf)).
\]

The three expressions add up to zero.

**Problem 2: Chapter 1, #22 (5pts)**

Let $\gamma(t)$ be an integral curve for a vector field $X$ on $M$. Show that if $\gamma'(t) = 0$ for some $t$, then $\gamma$ is a constant map.

Suppose $\gamma: (a, b) \to M$, $\gamma'(t_0) = 0$ for some $t_0 \in (a, b)$, and $\gamma(t_0) = p$. Since $\gamma$ is an integral curve for $X$,

\[X(p) = X(\gamma(t_0)) = \gamma'(t_0) = 0.\]

Let $\beta: (a, b) \to M$ be the curve defined by $\beta(t) = p$ for all $t \in (a, b)$. Then,

\[\beta(t_0) = p \quad \text{and} \quad \beta'(t) = 0 = X(p) = X(\beta(t)) \quad \forall t \in (a, b).\]

We also have

\[\gamma(t_0) = p \quad \text{and} \quad \gamma'(t) = X(\gamma(t)) \quad \forall t \in (a, b).\]

By the uniqueness theorem for first-order ODEs, or Theorem 1.48, $\beta = \gamma$, i.e. $\gamma$ is a constant map.

**Problem 3: Chapter 1, #17 (5pts)**

Show that any smooth vector field on a compact manifold is complete.

Suppose $M$ is a compact $m$-manifold, $X$ is a smooth vector field on $M$, and $\gamma: (a, b) \to M$ is a maximal integral curve for $X$. Thus, $a < 0$ and $b > 0$. We need to show that $(a, b) = \mathbb{R}$.

Suppose $b \in \mathbb{R}$. Choose a sequence $t_n \in (a, b)$ converging to $b$. Since $M$ is a compact, a subsequence converges to a point $p \in M$. By (3) of Theorem 1.48, there exists $\epsilon \in (0, |a|)$ and a neighborhood $U$ of $p$ in $M$ such that the flow

\[(-\epsilon, \epsilon) \times U \to M, \quad (t, q) \mapsto X_t(q),\]

is well-defined. Choose $t_n \in (a, b)$ such that $b - t_n < \epsilon$ and $\gamma(t_n) \in U$. Let

\[\beta: (-\epsilon, \epsilon) \to M\]
be the integral curve for $X$ such that $\beta(0) = \gamma(t_n)$. Define

$$\alpha: (a, t_n + \epsilon) \rightarrow M \quad \text{by} \quad \alpha(t) = \begin{cases} \gamma(t), & \text{if } t \in (a, b); \\ \beta(t-t_n), & \text{if } t \in (t_n - \epsilon, t_n + \epsilon). \end{cases}$$

Let $\tilde{\gamma}(t) = \gamma(t+t_n)$ for $t \in (-\epsilon, b-t_n)$. Since

$$\tilde{\gamma}(0) = \beta(0), \quad \tilde{\gamma}'(t) = X(\tilde{\gamma}(t)), \quad \text{and} \quad \beta'(t) = X(\beta(t)),$$

$\tilde{\gamma} = \beta$ on $(-\epsilon, b-t_n)$ by the uniqueness of integral curves. Thus, $\alpha$ is well-defined. Furthermore,

$$\alpha'(t) = \begin{cases} \gamma'(t), & \text{if } t \in (a, b) \\ \beta'(t-t_n), & \text{if } t \in (t_n - \epsilon, t_n + \epsilon) \end{cases} = \begin{cases} X(\gamma(t)), & \text{if } t \in (a, b) \\ X(\beta(t-t_n)), & \text{if } t \in (t_n - \epsilon, t_n + \epsilon) \end{cases} = X(\alpha(t)),$$

i.e. $\alpha$ is an integral curve for $X$. Since $t_n + \epsilon > b$ and $\alpha|_{(a,b)} = \gamma$, we conclude that $\alpha$ is an integral curve for $X$ extending $\gamma$. Thus, $\gamma$ is not maximal unless $b = \infty$. The proof that $a = -\infty$ is similar (or apply the conclusion to the vector field $-X$).

**Problem 4 (5pts)**

Let $V$ be the vector field on $\mathbb{R}^3$ given by

$$V(x, y, z) = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Explicitly describe and sketch the flow of $V$.

The integral curves for this vector fields are the solutions of the system

$$x'(t) = y(t), \quad y'(t) = -x(t), \quad z'(t) = 1.$$

The first pair of equations is independent of the third; its solutions are

$$(x(t), y(t)) = (x_0 + iy_0)e^{-it} \in \mathbb{C}.$$  

The corresponding curve goes around a circle centered at the origin *clockwise* at the unit angular speed. The solution of the third equation is $z(t) = z_0 + t$. Thus, the flow for $X$ is given by

$$X_t: \mathbb{R}^3 \rightarrow \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}, \quad X_t(x, y, z) = ((x+iy)e^{-it}, z+t),$$

i.e. the flow rotates clockwise around the vertical axis at the unit angular speed and climbs up at the unit angular speed (the vertical axis itself simply moves up).
Problem 5 (10pts)

Suppose $X$ and $Y$ are smooth vector fields on a manifold $M$. Show that for every $p \in M$ and $f \in C^\infty(M)$,

$$\lim_{s,t \to 0} \frac{f(Y_{-s}(X_{-t}(Y_s(X_t(p)))) - f(p)}{st} = [X,Y]_p f \in \mathbb{R}.$$ 

Do not forget to explain why the limit exists.

Note: This means that the extent to which the flows $\{X_t\}$ of $X$ and $\{Y_s\}$ of $Y$ do not commute (i.e. the rate of change in the "difference" between $Y_s \circ X_t$ and $X_t \circ Y_s$) is measured by $[X,Y]$.

By (d) of Theorem 1.48, we can choose a neighborhood $U$ of $p$ and $\epsilon > 0$ such that the maps

$$(-\epsilon, \epsilon) \times U \to M, \quad (z, q) \to X_z(q), \quad (-\epsilon, \epsilon)^2 \times U \to M, \quad (w, z, q) \to Y_w(X_z(q)),$$

$$(-\epsilon, \epsilon)^3 \times U \to M, \quad (v, w, z, q) \to Y_v(Y_w(X_z(q))), \quad \text{and}$$

$$(-\epsilon, \epsilon)^4 \times U \to M, \quad (u, v, w, z, q) \to Y_u(Y_v(Y_w(X_z(q)))),$$

are defined and smooth. Define

$$K: (-\epsilon, \epsilon)^4 \to \mathbb{R} \quad \text{and} \quad H: (-\epsilon, \epsilon)^2 \to \mathbb{R} \quad \text{by}$$

$$K(u, v, w, z) = f(Y_u(Y_v(Y_w(X_z(p)))))) - f(p) \quad \text{and} \quad H(s, t) = K(-s, -t, s, t).$$

Since $K$ is a composition of smooth functions, $K$ is smooth. Since $H$ is a composition of smooth functions, $H$ is smooth. Furthermore,

$$X_0 = Y_0 = \text{id}_X, \quad Y_s \circ Y_s = \text{id}_{\text{Dom}Y}, \quad X_{-t} \circ X_t = \text{id}_{\text{Dom}X} \implies H(s, 0) = H(0, t) = 0$$

for all $s$ and $t$. Thus\(^1\), the limit in the statement of the problem exists and equals to the mixed second partial derivative of $H$:

$$\lim_{s,t \to 0} \frac{f(Y_{-s}(X_{-t}(Y_s(X_t(p))))) - f(p)}{st} = \lim_{s,t \to 0} \frac{H(s, t) - H(0, 0)}{st} = \frac{\partial^2 H}{\partial s \partial t}(0,0).$$

\(^1\)this follows, for example, from Lemma 3.5 in Lecture Notes applied twice
On the other hand, by the Chain Rule,
\[
\frac{\partial^2 H}{\partial s \partial t}\bigg|_{(0,0)} = \frac{\partial^2 K}{\partial u \partial v}\bigg|_{(0,0,0,0)} - \frac{\partial^2 K}{\partial u \partial z}\bigg|_{(0,0,0,0)} - \frac{\partial^2 K}{\partial v \partial w}\bigg|_{(0,0,0,0)} + \frac{\partial^2 K}{\partial w \partial z}\bigg|_{(0,0,0,0)}.
\]
Note that
\[
\frac{\partial^2 K}{\partial u \partial v}\bigg|_{(0,0,0,0)} = \frac{\partial}{\partial v}\left(\frac{\partial}{\partial u}(Y_u(X_v(p)))\bigg|_{u=0}\right) = \frac{\partial}{\partial v}\left(\frac{dX_v(p)}{\partial u}Y_u(X_v(p))\bigg|_{u=0}\right)\bigg|_{v=0} = \frac{\partial}{\partial v}\left(Y_{f}(X_v(p))\bigg|_{v=0} = d_{p}\{Y_{f}\}(X) = X_{p}(Y_{f}).
\]
Similarly,
\[
\frac{\partial^2 K}{\partial u \partial z}\bigg|_{(0,0,0,0)} = X_{p}(Y_{f}), \quad \frac{\partial^2 K}{\partial v \partial w}\bigg|_{(0,0,0,0)} = Y_{p}(X_{f}), \quad \frac{\partial^2 K}{\partial w \partial z}\bigg|_{(0,0,0,0)} = X_{p}(Y_{f}).
\]
Putting these together, we conclude that
\[
\lim_{s,t \to 0} \frac{f(Y_{s-t}(X_{e}(X_{t}(p)))) - f(p)}{s t} = \frac{\partial^2 H}{\partial s \partial t}\bigg|_{(0,0)} = X_{p}(Y_{f}) - Y_{p}(X_{f}) = [X,Y]p.f.
\]

**Problem 6 (10pts)**

Let \( U \) and \( V \) be the vector fields on \( \mathbb{R}^3 \) given by
\[
U(x,y,z) = \frac{\partial}{\partial x} \quad \text{and} \quad V(x,y,z) = F(x,y,z)\frac{\partial}{\partial y} + G(x,y,z)\frac{\partial}{\partial z},
\]
where \( F \) and \( G \) are smooth functions on \( \mathbb{R}^3 \). Show that there exists a proper\(^2\) foliation of \( \mathbb{R}^3 \) by 2-dimensional embedded submanifolds such that the vector fields \( U \) and \( V \) everywhere span the tangent spaces of these submanifolds if and only if
\[
F(x,y,z) = f(y,z)e^{h(x,y,z)} \quad \text{and} \quad G(x,y,z) = g(y,z)e^{h(x,y,z)}
\]
for some \( f, g \in C^\infty(\mathbb{R}^2) \) and \( h \in C^\infty(\mathbb{R}^3) \) such that \( (f,g) \) does not vanish on \( \mathbb{R}^2 \).

If at every point of \( \mathbb{R}^3 \) the vector fields \( U \) and \( V \) span the tangent space of a 2-dimensional submanifold, then their span is two-dimensional, i.e. \( (F,G) \) does not vanish. If this is the case, by Frobenius Theorem there exists an integral submanifold for the distribution \( D \subset T\mathbb{R}^3 \) spanned by \( U \) and \( V \) through every point of \( \mathbb{R}^3 \) if and only if the vector field
\[
[U,V] = F_x \frac{\partial}{\partial y} + G_x \frac{\partial}{\partial z}
\]
lies in the span of \( U \) and \( V \) over \( C^\infty(\mathbb{R}^3) \). This is the case if and only if there exists \( \lambda \in C^\infty(\mathbb{R}^3) \) such that
\[
[U,V] = \lambda V \quad \iff \quad F_x = \lambda F, \quad G_x = \lambda G
\]
\[
\iff \quad F(x,y,z) = f(y,z)e^{h(x,y,z)}, \quad G(x,y,z) = g(y,z)e^{h(x,y,z)},
\]
\(^2\)in the sense of Definition 10.4 in Lecture Notes
where \( h \in C^\infty(\mathbb{R}^3) \) is such that \( h_x = \lambda \) and \( f, g \in C^\infty(\mathbb{R}^2) \) are such that \( (f, g) \) does not vanish on \( \mathbb{R}^2 \) (so that \( V \) does not vanish).

If the above is the case, the maximal connected integral submanifolds for the distribution \( \mathcal{D} \) spanned by \( U \) and \( V \) partition \( \mathbb{R}^3 \). We will show that all such submanifolds are embedded. Since \( e^h \) does not vanish, \( \mathcal{D} \) is spanned by the vector fields

\[
U(x, y, z) = \frac{\partial}{\partial x} \quad \text{and} \quad W(x, y, z) = f(y, z) \frac{\partial}{\partial y} + g(y, z) \frac{\partial}{\partial z}.
\]

Let \( \gamma: (a, b) \to \mathbb{R}^2 \) be a maximal integral curve for the vector field

\[
\tilde{W}(y, z) = f(y, z) \frac{\partial}{\partial y} + g(y, z) \frac{\partial}{\partial z}.
\]

Since \( (f, g) \) does not vanish on \( \mathbb{R}^2 \) and \( \gamma'(t) = \tilde{W}(\gamma(t)) \) is a maximal connected integral submanifold for the distribution \( \tilde{\mathcal{D}} \) on \( \mathbb{R}^2 \) spanned by \( \tilde{W} \). Furthermore,

\[
\psi = \text{id} \times \gamma: \mathbb{R} \times (a, b) \to \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}^2
\]

is a maximal connected integral submanifold for \( \mathcal{D} \) and every maximal connected submanifold for \( \mathcal{D} \) has this form. It is an integral submanifold for \( \mathcal{D} \) because

\[
d\psi\big|_{(s,t)} \frac{\partial}{\partial s} = \frac{\partial}{\partial s} \times 0 = U\big(\psi(s, t)\big),
\]

\[
d\psi\big|_{(s,t)} \frac{\partial}{\partial t} = (0, \gamma'(t)) = (0, \tilde{W}(\gamma(t))) = W\big(\psi(s, t)\big),
\]

if \((s,t)\) are the standard coordinates on \( \mathbb{R} \times (a, b) \). Since the maximal integral curves \( \gamma \) for \( W \) partition \( \mathbb{R}^2 \), the images of the maps \( \text{id} \times \gamma \) partition \( \mathbb{R}^3 \). Thus, each map \( \text{id} \times \gamma \) must be a maximal connected submanifold for \( \mathcal{D} \). In the following paragraph, we show that every integral curve \( \gamma \) for \( \tilde{W} \) must be embedded in \( \mathbb{R}^2 \). This implies that every maximal connected integral submanifold \( \text{id} \times \gamma \) for \( \mathcal{D} \) is embedded in \( \mathbb{R}^3 \).

Suppose \( \gamma: (a, b) \to \mathbb{R}^2 \) is a maximal integral curve for \( \tilde{W} \) and \( t_0 \in (a, b) \). By Proposition 1.53, we can choose a coordinate chart

\[
\varphi = (x_1, x_2): (\mathcal{U}, \gamma(t_0)) \to (\mathbb{R}^2, 0)
\]

and a neighborhood \((c, d)\) of \( t_0 \) in \((a, b)\) such that

\[
[-2, 2] \times [-2, 2] \subset \varphi(\mathcal{U}), \quad W_{|\mathcal{U}} = \frac{\partial}{\partial x_1}_{\mathcal{U}}, \quad \text{and} \quad \gamma_{|(c, d)}: (c, d) \to \varphi^{-1}(0 \times (-2, 2)) \quad (1)
\]

is a diffeomorphism. The middle condition implies that

\[
\text{Im} \gamma \cap \varphi^{-1}((-2, 2) \times (-2, 2))
\]

is a union of horizontal slices \( \varphi^{-1}((-2, 2) \times y) \) with \( y \in S_\gamma \), where \( S_\gamma \) is a subset of \((-2, 2)\). To show that \( \gamma \) is an embedding, we show that there exists \( \epsilon > 0 \) such that

\[
S_\gamma \cap (-\epsilon, \epsilon) = \{0\}.
\]

\[\text{In such a case, if } (c', d') \subset (c, d) \text{ is a basis element around } t_0 \in (a, b), \text{ then}
\]
\[
\gamma((c', d')) = \gamma((c', d')) \times 0 = \gamma((c', d')) \cap ((\gamma(c'), \gamma(d')) \times (-\epsilon, \epsilon)),
\]

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Suppose not, i.e. there exists a sequence $t_k \in (a, b)$ converging to either $a$ or $b$ such that $\gamma(t_k) = 0 \times y_k$ with $y_k \in S_\gamma$ converging to $0 \in \mathbb{R}$. By taking a subsequence and by symmetry, it is sufficient to assume that $t_k \to b$ and $y_k \in \mathbb{R}^+$. We can then choose $t_1, t_2 \in (t_0, b)$ with $t_1 < t_2$ so that

$$\varphi(\gamma(t_1)) = 0 \times y_1, \quad \varphi(\gamma(t_2)) = 0 \times y_2 \quad \text{s.t.}$$

$$0 < y_2 < y_1 < 2, \quad (t_0, t_1) \cap \gamma^{-1}(\varphi^{-1}(0 \times (0, y_1))) = \emptyset, \quad (t_1, t_2) \cap \gamma^{-1}(\varphi^{-1}(0 \times (0, y_1))) = \emptyset.$$

In other words, $t_1 \in (t_0, b)$ is the smallest number so that $\varphi(\gamma(t_1)) \in 0 \times (0, 2)$ and $t_2 \in (t_1, b)$ is the smallest number so that $\varphi(\gamma(t_2)) \in 0 \times (0, y_1)$.

By the middle condition in eq1,

$$t_1' \equiv \gamma^{-1}(\varphi^{-1}(1 \times y_1)) \in (t_1, t_2), \quad t_2' \equiv \gamma^{-1}(\varphi^{-1}((-1) \times y_2)) \in (t_1', t_2),$$

i.e. the flow is to the right in each slice as indicated in the diagram.

Since $\gamma$ is an integral curve, it has no self-intersections. Thus,

$$\mathcal{C} \equiv \gamma((t_0, t_1)) \cup \varphi^{-1}(0 \times [0, y_1])$$

is a simple closed curve in $\mathbb{R}^2$ since

$$(t_0, t_1) \cap \gamma^{-1}(\varphi^{-1}(0 \times (0, y_1))) = \emptyset.$$

Let $\ell$ be the straight line segment between $\varphi(\gamma(t_1'))$ and $\varphi(\gamma(t_2'))$ in $\mathbb{R}^2$. Since the curve $\varphi^{-1}(\ell)$ intersects the simple closed curve $\mathcal{C}$ exactly once, one of its endpoints, i.e. $\gamma(t_1')$ or $\gamma(t_2')$, must lie inside of $\mathcal{C}$ and the other outside. Thus, the curve $\gamma((t_1', t_2'))$ must intersect $\mathcal{C}$ at least once. Since

$$\gamma((t_1', t_2')) \cap \varphi^{-1}(0 \times [0, y_1]) \subset \gamma((t_1, t_2)) \cap \varphi^{-1}(0 \times [0, y_1]) = \emptyset$$

and $\gamma((t_1', t_2')) \cap \mathcal{C} \neq \emptyset$, we conclude that

$$\gamma((t_1', t_2')) \cap \gamma((t_0, t_1)) \neq \emptyset.$$

However, this is impossible as well, since $t_1 < t_1'$ and an integral curve cannot intersect itself.

Remark: The above argument implies that if $\mathcal{D}$ is any distribution on $\mathbb{R}^2$, every connected integral submanifold for $\mathcal{D}$ is embedded. This is not the case for other manifolds, including $\mathbb{R}^3$ and $T^2$ (see Chapter 1, #21, p51).

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i.e. $\gamma$ takes open subsets of $(a, b)$ to sets in $\text{Im } \gamma$ that are open with respect to the topology $\text{Im } \gamma$ inherits as a subspace of $\mathbb{R}^2$. So, $\gamma$ is an embedding.