MAT 531: Topology & Geometry, II  
Spring 2011

Problem Set 3  
Due on Thursday, 2/24, in class

1. Chapter 1, #5 (p50)

2. Show that the tangent bundle $TM$ of a smooth $n$-manifold $M$ is a real vector bundle of rank $n$ over $M$. What is its transition data?

3. Show that the tangent bundle $TS^1$ of $S^1$, defined as in 1.25 (p19), is isomorphic to the trivial real line bundle over $S^1$. Hint: Use Lemma 8.5 in Lecture Notes.

4. Suppose that $f : X \to M$ is a smooth map and $\pi : V \to M$ is a smooth vector bundle. The pullback of $V$ by $f$, $\pi^1 : f^*V \to X$, is the vector bundle defined by taking
   \[ f^*V = \{(x,v) \in X \times V : f(x) = \pi(v)\} \subset X \times V. \]
   Show that $f^*V$ is indeed a smooth submanifold of $X \times V$.

5. Show that the tautological line bundle $\gamma_n \to \mathbb{C}P^n$ is indeed a complex line bundle (describe its trivializations). What is its transition data? Why is it non-trivial for $n \geq 1$? (not isomorphic to $\mathbb{C}P^n \times \mathbb{C} \to \mathbb{C}P^n$ as line bundle over $\mathbb{C}P^n$). Hint: See proof of Lemma 8.4 in Lecture Notes.

6. Suppose $k < n$. Show that the map
   \[ \iota : \mathbb{C}P^k \to \mathbb{C}P^n, \quad [X_0, \ldots, X_k] \mapsto [X_0, \ldots, X_k, 0, \ldots, 0], \]
   is a complex embedding (i.e. a smooth embedding that induces holomorphic maps between the charts that determine the complex structures on $\mathbb{C}P^k$ and $\mathbb{C}P^n$). Show that the normal bundle to this immersion, $\mathcal{N}_\iota$, is isomorphic to
   \[ (n-k)\gamma_k^* \equiv \underbrace{\gamma_k^* \oplus \cdots \oplus \gamma_k^*}_{n-k}, \]
   where $\gamma_k \to \mathbb{C}P^k$ is the tautological line bundle (isomorphic as complex line bundles).

7. Let $\Lambda^n_T \mathbb{C}P^n \to \mathbb{C}P^n$ be the top exterior power of the vector bundle $T \mathbb{C}P^n$ taken over $\mathbb{C}$. Show that $\Lambda^n_T \mathbb{C}P^n$ is isomorphic to the line bundle
   \[ \gamma_n^* \otimes (n+1) \equiv \underbrace{\gamma_n^* \otimes \cdots \otimes \gamma_n^*}_{n+1}, \]
   where $\gamma_n \to \mathbb{C}P^n$ is the tautological line bundle (isomorphic as complex line bundles).

*Hint* on the next page
**Hint for 6 and 7:** There are a number of ways of doing these, including:

(i) construct an isomorphism between the two line bundles;

(ii) show that there exists a short exact sequence of vector bundles

\[ 0 \rightarrow \mathbb{C}P^n \times \mathbb{C} \rightarrow (n+1)\gamma_n^* \rightarrow T\mathbb{C}P^n \rightarrow 0 \]

and this implies the claim (exact means that at each position the kernel of the outgoing map equals to the image of the incoming map over every point of \( M \));

(iii) use Problems PS1-3b and 2 and 5 above to determine transition data for the two bundles. However, you will need to modify trivializations for one of the line bundles in Problem 7 to arrive at the same transition data;

(iv) show that there exists a holomorphic diffeomorphism between \((n-k)\gamma_k^*\) and a neighborhood of \( \iota(\mathbb{C}P^k) \) in \( \mathbb{C}P^n \) and that this implies the claim in Problem 6.