MAT 531: Topology&Geometry, II
Spring 2011

Problem Set 11
Due on Thursday, 5/12, in class

1. Let M and N be compact oriented connected n-manifolds. If $f\colon M \longrightarrow N$ is a smooth map, the degree of f is the number $\deg f \in \mathbb{R}$ such that

$$\int_M f^*\omega = (\deg f)\cdot \int_N \omega \qquad \forall\, \omega \in E^n(N).$$

   This number is well-defined (i.e. exists) and is an integer (you should be able to prove these two claims, but you can assume them for this problem).

   (a) Show that if $f: M \longrightarrow N$ and $g: N \longrightarrow X$ are smooth maps between compact oriented connected n-manifolds, then

$$\deg(g\circ f) = (\deg g)\cdot(\deg f).$$

   (b) Show that if $f: M \longrightarrow N$ is a covering projection between compact oriented connected n-manifolds, then $\deg f$ is the degree of f as a covering map (i.e. the number of elements in each fiber).

   (c) Show that if $f\colon M \longrightarrow N$ is a smooth map of degree one between compact oriented connected n-manifolds, then

$$f_*\colon \pi_1(M) \longrightarrow \pi_1(N)$$

   is surjective.

   Note: The last part uses a fact that has been stated previously and will be proved in class on Tuesday, 5/10. You can nearly complete the solution to this problem without this fact.

2. State and prove a Mayer-Vietoris theorem for compactly supported cohomology.
   Note: The restriction of a compactly supported form to an open subset need not be compactly supported. You do not need anything from the lecture on Tuesday, 5/10, to do this problem.

3. Let M be an oriented n-manifold, possibly non-compact.

   (a) Show that the pairing

$$H^*_{\mathrm{de\,R}}(M)\otimes H^*_{\mathrm{de\,R};c}(M) \longrightarrow \mathbb{R}, \qquad [\alpha]\otimes[\beta] \longrightarrow \int_M \alpha\wedge\beta,$$

   is well-defined.

   (b) Show that the above pairing is nondegenerate if $M = \mathbb{R}^n$.

   (c) Suppose that M admits a cover $\{U_i\}_{i=1,\ldots,m}$ such that every intersection $U_{i_1}\cap\ldots\cap U_{i_k}$ is either empty or diffeomorphic to $\mathbb{R}^n$. Show that the above pairing is nondegenerate.