MAT 531: Topology&Geometry, II
Spring 2011

Solutions to Problem Set 10

Problem 1: Chapter 6, #6 (5pts)

Derive explicit formulas for d, $*$, δ, and Δ in Euclidean space.

Let $x_1, \ldots, x_n$ denote the standard coordinate functions on $\mathbb{R}^n$. It is sufficient to describe the action of these linear operators on forms

$$\alpha = f\, dy_1 \wedge \ldots \wedge dy_p,$$

where $f \in C^\infty(\mathbb{R}^n)$ and $y_1, \ldots, y_n$ is a permutation of the coordinates $x_1, \ldots, x_n$ so that

$$dy_1 \wedge \ldots \wedge dy_n = dx_1 \wedge \ldots \wedge dx_n.$$

By Section 2.20 and Exercise 13 in Chapter 2,

$$d\alpha = df \wedge dy_1 \wedge \ldots \wedge dy_p = \sum_{j=p+1}^{j=n} \frac{\partial f}{\partial y_j} dy_j \wedge dy_1 \wedge \ldots \wedge dy_p = (-1)^p \sum_{j=p+1}^{j=n} \frac{\partial f}{\partial y_j} dy_1 \wedge \ldots \wedge dy_p \wedge dy_j;$$
$$*\alpha = f\, dy_{p+1} \wedge \ldots \wedge dy_n.$$

Thus, by Section 6.1,

$$\begin{aligned}
\delta\alpha &= (-1)^{n(p+1)+1} * d * \alpha = (-1)^{n(p+1)+1} * d\big(f\, dy_{p+1} \wedge \ldots \wedge dy_n\big) \\
&= (-1)^{n(p+1)+1} * \sum_{i=1}^{i=p} \frac{\partial f}{\partial y_i} dy_i \wedge dy_{p+1} \wedge \ldots \wedge dy_n \\
&= (-1)^{n(p+1)+1} \sum_{i=1}^{i=p} (-1)^{(n-p)(p-1)+(i-1)} \frac{\partial f}{\partial y_i} dy_1 \wedge \ldots \wedge \widehat{dy_i} \wedge \ldots \wedge dy_p \\
&= \sum_{i=1}^{i=p} (-1)^i \frac{\partial f}{\partial y_i} dy_1 \wedge \ldots \wedge \widehat{dy_i} \wedge \ldots \wedge dy_p.
\end{aligned}$$

From this, we find that

$$\begin{aligned}
\Delta\alpha = d\delta\alpha + \delta d\alpha &= d \sum_{i=1}^{i=p} (-1)^i \frac{\partial f}{\partial y_i} dy_1 \wedge \ldots \wedge \widehat{dy_i} \wedge \ldots \wedge dy_p + (-1)^p \delta \sum_{j=p+1}^{j=n} \frac{\partial f}{\partial y_j} dy_1 \wedge \ldots \wedge dy_p \wedge dy_j \\
&= \sum_{i=1}^{i=p} (-1)^i \bigg((-1)^{i-1} \frac{\partial^2 f}{\partial y_i^2} dy_1 \wedge \ldots \wedge dy_p + (-1)^{p-1} \sum_{j=p+1}^{j=n} \frac{\partial^2 f}{\partial y_i \partial y_j} dy_1 \wedge \ldots \wedge \widehat{dy_i} \wedge \ldots \wedge dy_p \wedge dy_j \bigg) \\
&\quad + (-1)^p \sum_{j=p+1}^{j=n} \bigg((-1)^{p+1} \frac{\partial^2 f}{\partial y_j^2} dy_1 \wedge \ldots \wedge dy_p + \sum_{i=1}^{i=p} (-1)^i \frac{\partial^2 f}{\partial y_i \partial y_j} dy_1 \wedge \ldots \wedge \widehat{dy_i} \wedge \ldots \wedge dy_p \wedge dy_j \bigg) \\
&= -\sum_{i=1}^{i=n} \frac{\partial^2 f}{\partial y_i^2}\, dy_1 \wedge \ldots \wedge dy_p.
\end{aligned}$$

Problem 2 (5pts)

Suppose M is a compact Riemannian manifold. If M is not orientable, the Hodge $*$-operator is defined only up to sign. However, as the definition of δ in Section 6.1 involves two $*$'s, the linear operator δ is well-defined. Show that δ is the adjoint of d whether or not M is oriented.

The oriented case is dealt with in Proposition 6.2. Thus, it is sufficient to assume that M is connected and non-orientable. Let $\pi\colon \tilde{M} \longrightarrow M$ be the orientable double cover of M; see solution to Problem 5 on the 06 midterm. The Riemannian metric on M (or inner-product in $TM \longrightarrow M$) induces via the projection map $d\pi$ a Riemannian metric on $\tilde{M}$ so that

$$(d\pi_x)^*\colon \Lambda^p T^*_{\pi(x)}M \longrightarrow \Lambda^p T^*_x\tilde{M}$$

is an isometry for all p and $x\in\tilde{M}$. In particular,

$$\langle\alpha,\beta\rangle = \langle\pi^*\alpha,\pi^*\beta\rangle \quad \forall\ \alpha,\beta\in\Lambda^pT^*_mM \qquad \Longrightarrow \qquad \langle\langle\alpha,\beta\rangle\rangle = \frac{1}{2}\langle\langle\pi^*\alpha,\pi^*\beta\rangle\rangle \quad \forall\ \alpha,\beta\in E^p(M),$$

since π is a double cover. Choose an orientation on $\tilde{M}$ so that $*$ is defined on $\tilde{M}$. Given $x\in\tilde{M}$, we can compute

$$(\delta\alpha)_{\pi(x)} \equiv (-1)^{n(p+1)+1}(*d*\alpha)_{\pi(x)}$$

by using the orientation on a small neighborhood of $\pi(x)$ induced from the orientation on a small neighborhood of x via $d\pi$. Then, π^* commutes with $*$ near x and thus with δ everywhere. Therefore, by the above and Proposition 6.2

$$\begin{aligned}\langle\langle d\alpha,\beta\rangle\rangle = \frac{1}{2}\langle\langle\pi^*d\alpha,\pi^*\beta\rangle\rangle &= \frac{1}{2}\langle\langle d\,\pi^*\alpha,\pi^*\beta\rangle\rangle \\ &= \frac{1}{2}\langle\langle\pi^*\alpha,\delta\,\pi^*\beta\rangle\rangle = \frac{1}{2}\langle\langle\pi^*\alpha,\pi^*\delta\beta\rangle\rangle = \langle\langle\pi^*\alpha,\delta\beta\rangle\rangle\end{aligned}$$

for all $\alpha\in E^{p-1}(M)$ and $\beta\in E^p(M)$. Thus, $\delta=d^*$ on M.

Problem 3 (5pts)

Suppose M is a compact connected non-orientable n-manifold. Show that $H^n_{\mathrm{de\,R}}(M)=0$.
Remark: In fact, if M is a *non*-compact connected n-manifold, orientable or not, $H^n_{\mathrm{de\,R}}(M)=0$.

Let $\pi\colon\tilde{M}\longrightarrow M$ be the orientable double cover of M and let G be the group of covering transformations of π. This group consists of two elements; let g be the non-trivial element. By Problem 5a on PS7,

$$H^n_{\mathrm{de\,R}}(M) = H^n_{\mathrm{de\,R}}(\tilde{M})^G = \big\{[\omega]\in H^n_{\mathrm{de\,R}}(\tilde{M})\colon [g^*\omega]=[\omega]\big\}.$$

On the other hand, since $\tilde{M}$ is compact, connected, and orientable, $H^n_{\mathrm{de\,R}}(\tilde{M})\approx\mathbb{R}$ by Corollary 6.13. Thus, it is sufficient to find a single element $[\omega]\in H^n_{\mathrm{de\,R}}(\tilde{M})$ such that $[g^*\omega]\neq[\omega]$.

Let $\omega\in E^n(M)$ be a nowhere-zero top form on $\tilde{M}$. Since M is not orientable, $g^*\omega$ belongs to the opposite orientation for $\tilde{M}$; see solutions to Problem 6 on PS6. In other words, $g^*\omega=f\cdot\omega$ for some smooth function $f\colon\tilde{M}\longrightarrow\mathbb{R}^-$. Thus,

$$\int_{\tilde{M}} g^*\omega = \int_{\tilde{M}} f\cdot\omega \neq \int_{\tilde{M}}\omega \qquad \Longrightarrow \qquad [g^*\omega]\neq[\omega]\in H^n_{\mathrm{de\,R}}(\tilde{M})\,;$$

the two integrals above are not equal because they have opposite signs.

Problem 4: Chapter 6, #16 (30pts)

Suppose M is a compact Riemannian manifold and $\Delta : E^p(M) \longrightarrow E^p(M)$ is the corresponding Laplacian. Show that

(a) *all eigenvalues of Δ are non-negative;*

(d) *eigenfunctions corresponding to distinct eigenvalues are orthogonal;*

(b) *eigenspaces of Δ are finite-dimensional;*

(c) *the set of eigenvalues of Δ has no limit point;*

(e) *Δ has a positive eigenvalue;*

(f) *Δ has infinitely many positive eigenvalues;*

(g) *the linear span of eigenfunctions of Δ is L^2-dense in $E^p(M)$;*

(h) *the linear span of eigenfunctions of Δ is L^∞-dense in $E^p(M)$.*

(a) Suppose $\alpha \in E^p(M)$ is an eigenfunction of Δ with eigenvalue $\lambda \in \mathbb{R}$, i.e. $\alpha \neq 0$ and $\Delta\alpha = \lambda\alpha$. Since $\Delta = d^*d + dd^*$,

$$\begin{aligned} \lambda|\alpha|^2 = \lambda\langle\langle\alpha,\alpha\rangle\rangle = \langle\langle\lambda\alpha,\alpha\rangle\rangle = \langle\langle\Delta\alpha,\alpha\rangle\rangle &= \langle\langle d^*d\alpha,\alpha\rangle\rangle + \langle\langle dd^*\alpha,\alpha\rangle\rangle \\ &= \langle\langle d\alpha,d\alpha\rangle\rangle + \langle\langle d^*\alpha,d^*\alpha\rangle\rangle = |d\alpha|^2 + |d^*\alpha|^2 \geq 0. \end{aligned}$$

Since $|\alpha|^2 > 0$, it follows that $\lambda \geq 0$.

(d) Suppose $\alpha_1, \alpha_2 \in E^p(M)$ are eigenfunctions of Δ with eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$. Since $\Delta^* = \Delta$,

$$\begin{aligned} \lambda_1\langle\langle\alpha_1,\alpha_2\rangle\rangle = \langle\langle\lambda_1\alpha_1,\alpha_2\rangle\rangle &= \langle\langle\Delta\alpha_1,\alpha_2\rangle\rangle \\ &= \langle\langle\alpha_1,\Delta\alpha_2\rangle\rangle = \langle\langle\alpha_1,\lambda_2\alpha_2\rangle\rangle = \lambda_2\langle\langle\alpha_1,\alpha_2\rangle\rangle. \end{aligned}$$

Thus, if $\lambda_1 \neq \lambda_2$, $\langle\langle\alpha_1,\alpha_2\rangle\rangle = 0$, i.e. eigenspaces with different eigenvalues are orthogonal.

(b) Suppose not, i.e. there exists an orthonormal sequence $\alpha_1, \alpha_2, \ldots \in E^p(M)$ of eigenfunctions of Δ with eigenvalue $\lambda \in \mathbb{R}$. Then,

$$\|\alpha_n\| = 1 \qquad \text{and} \qquad \|\Delta\alpha_n\| = \|\lambda\alpha_n\| = \lambda\|\alpha_n\| = \lambda.$$

By Theorem 6.6, the sequence $\{\alpha_n\}$ contains a Cauchy subsequence. However, this is impossible, since it consists of orthonormal elements.

(c) Suppose not, i.e. there exists a sequence of distinct eigenvalues λ_n of Δ that converges to some $\lambda \in \mathbb{R}$. By part (a), it can be assumed that $0 \leq \lambda_n \leq \lambda+1$. Let $\alpha_n \in E^p(M)$ be a normal eigenfunction of Δ with eigenvalue λ_n. Then,

$$\|\alpha_n\| = 1 \qquad \text{and} \qquad \|\Delta\alpha_n\| = \|\lambda_n\alpha_n\| = \lambda_n\|\alpha_n\| \leq \lambda+1.$$

By Theorem 6.6, the sequence $\{\alpha_n\}$ contains a Cauchy subsequence. However, this is impossible, since it consists of orthonormal elements by part (d).

(e) Let $(\mathcal{H}^p)^\perp$ be the orthogonal complement of $\mathcal{H}^p \equiv \ker \Delta_p$ in $E^p(M)$ and let

$$G\colon E^p(M) \longrightarrow (\mathcal{H}^p)^\perp$$

be the Green's operator for Δ as in Section 6.9. In particular, G is a bounded linear operator by Theorem 6.6 (or eq4 in Section 6.8),

$$G\Delta\alpha = \Delta G\alpha \quad \forall\, \alpha \in (\mathcal{H}^p)^\perp \qquad \text{and} \qquad G\alpha = 0 \quad \forall\, \alpha \in \mathcal{H}^p \equiv \ker \Delta_p.$$

Furthermore, by the proof of part (a) and Theorem 6.8,

$$\langle\!\langle \Delta\alpha, \alpha \rangle\!\rangle \ge 0 \quad \forall\, \alpha \in E^p(M) \qquad \Longrightarrow \qquad \langle\!\langle \psi, G\psi \rangle\!\rangle \ge 0 \quad \forall\, \psi \in E^p(M). \tag{1}$$

Since G is bounded,

$$\eta \equiv \sup_{\varphi \in (\mathcal{H}^p)^\perp, \|\varphi\|=1} \|G\varphi\| < \infty.$$

Since G is nonzero on $(\mathcal{H}^p)^\perp$, $\eta > 0$. Furthermore,

$$\|G\varphi\| \le \eta \|\varphi\| \qquad \forall\, \varphi \in (\mathcal{H}^p)^\perp.$$

It will be shown that $1/\eta$ is an eigenvalue of Δ.

Let $\varphi_n \in (\mathcal{H}^p)^\perp$ be a sequence such that

$$\|\varphi_n\| = 1 \qquad \text{and} \qquad \lim_{n \longrightarrow \infty} \|G\varphi_n\| = \eta.$$

Then,

$$\|G\varphi_n\| \le \eta \|\varphi_n\| = \eta \qquad \text{and} \qquad \|\Delta(G\varphi_n)\| = \|\varphi_n\| = 1.$$

Thus, by Theorem 6.6, the sequence $\{G\varphi_n\}$ contains a Cauchy subsequence, which we still denote by $\{G\varphi_n\}$. Define

$$L\colon E^p(M) \longrightarrow \mathbb{R} \qquad \text{by} \qquad L(\beta) = \eta \cdot \lim_{n \longrightarrow \infty} \langle\!\langle G\varphi_n, \beta \rangle\!\rangle \quad \forall\, \beta \in E^p(M).$$

Since

$$\big| \langle\!\langle G\varphi_n, \beta \rangle\!\rangle - \langle\!\langle G\varphi_m, \beta \rangle\!\rangle \big| = \big| \langle\!\langle G\varphi_n - G\varphi_m, \beta \rangle\!\rangle \big| \le \|\beta\| \cdot \|G\varphi_n - G\varphi_m\|$$

and $\{G\varphi_n\}$ is Cauchy, the sequence $\langle\!\langle G\varphi_n, \beta \rangle\!\rangle$ is Cauchy in $\mathbb{R}$. Therefore, the limit above exists and thus L is a well-defined linear functional on $E^p(M)$. This functional is not zero. For, if m is sufficiently large,

$$\begin{gathered}
\eta/2 \le \|G\varphi_m\| \le \eta \qquad \text{and} \qquad \|G\varphi_m - G\varphi_n\| \le \eta/5 \quad \forall\, n \ge m \qquad \Longrightarrow \\
\big| \langle\!\langle G\varphi_n, G\varphi_m \rangle\!\rangle - \|G\varphi_m\|^2 \big| = \big| \langle\!\langle G\varphi_n - G\varphi_m, G\varphi_m \rangle\!\rangle \big| \le \big\| G\varphi_n - G\varphi_m \big\| \cdot \big\| G\varphi_m \big\| \le (\eta/5) \cdot \eta = \eta^2/5 \\
\Longrightarrow \qquad L(G\varphi_m) = \eta \cdot \lim_{n \longrightarrow \infty} \langle\!\langle G\varphi_n, G\varphi_m \rangle\!\rangle \ge \eta\big(\|G\varphi_m\|^2 - \eta^2/5\big) \ge \eta\big(\eta^2/4 - \eta^2/5\big) > 0.
\end{gathered}$$

The linear functional L is bounded, since

$$\big|L(\beta)\big| = \eta \cdot \lim_{n\longrightarrow\infty} \big|\langle\!\langle G\varphi_n, \beta\rangle\!\rangle\big| \le \eta \cdot \lim_{n\longrightarrow\infty} \|G\varphi_n\| \cdot \|\beta\| \le \eta \cdot \lim_{n\longrightarrow\infty} \eta\|\varphi_n\| \cdot \|\beta\| = \eta^2 \cdot \|\beta\|.$$

We show below that

$$L\big((\Delta-1/\eta)^*\beta\big) = L\big((\Delta-1/\eta)\beta\big) = 0 = \langle\!\langle 0, \beta\rangle\!\rangle \qquad \forall\ \beta\in E^p(M). \tag{2}$$

Thus, L is a weak solution of the equation $(\Delta-1/\eta)\omega=0$. Since Δ is an elliptic second-order differential operator by Section 6.35, so is $\Delta-1/\eta$. Thus, by the generalization of Theorem 6.6 stated in class on 4/22, there exists $\omega\in E^p(M)$ such that $L=L_\omega$, i.e.

$$L_\omega(\beta) \equiv \langle\!\langle \omega, \beta\rangle\!\rangle = \eta \cdot \lim_{n\longrightarrow\infty} \langle\!\langle G\varphi_n, \beta\rangle\!\rangle.$$

In particular, $(\Delta-1/\eta)\omega=0$. Since $L\not\equiv 0$, $\omega\neq 0$. Thus, $\omega\in E^p(M)$ is an eigenfunction of Δ with eigenvalue $1/\eta\in\mathbb{R}^+$.

Remark: If $\{\omega_n\}$ is a Cauchy sequence in an inner-product space A, $\omega\in A$, and

$$\lim_{n\longrightarrow\infty} \langle\!\langle \omega_n, \beta\rangle\!\rangle = \langle\!\langle \omega, \beta\rangle\!\rangle \qquad \forall\ \beta\in A, \tag{3}$$

then $\omega_n\longrightarrow\omega$. Given $\epsilon>0$, choose $m>0$ so that $\|\omega_m-\omega_n\|<\epsilon$ for all $n\ge m$. Then,

$$\|\omega-\omega_m\|^2 \le \big|\langle\!\langle \omega-\omega_n, \omega-\omega_m\rangle\!\rangle\big| + \big|\langle\!\langle \omega_n-\omega_m, \omega-\omega_m\rangle\!\rangle\big| \le \big|\langle\!\langle \omega-\omega_n, \omega-\omega_m\rangle\!\rangle\big| + \epsilon\|\omega-\omega_m\|.$$

By the above convergence assumption for $\beta=\omega-\omega_m$,

$$\lim_{n\longrightarrow\infty} \langle\!\langle \omega-\omega_n, \omega-\omega_m\rangle\!\rangle = 0 \qquad \Longrightarrow \qquad \|\omega-\omega_m\|^2 \le \epsilon\|\omega-\omega_m\| \qquad \Longrightarrow \qquad \|\omega-\omega_m\| \le \epsilon.$$

In our case, this implies that $\eta G\varphi_n\longrightarrow\omega$. The assumption that $\{\omega_n\}$ is a Cauchy sequence is required and does not follow from (3). For example, let

$$A = \ell_2 \equiv \Big\{(x_1,x_2,\ldots)\in\mathbb{R}^{\mathbb{Z}^+} : \sum_{i=1}^{\infty}|x_i|^2<\infty\Big\}, \qquad \langle\!\langle (x_i)_{i\in\mathbb{Z}^+}, (y_i)_{i\in\mathbb{Z}^+}\rangle\!\rangle = \sum_{i=1}^{\infty} x_i y_i\,,$$

and take $\omega_n=e_n\in\ell_2$ be the vector with the i-th coordinate 1 and the rest 0. Then,

$$\lim_{n\longrightarrow\infty} \langle\!\langle \omega_n, \beta\rangle\!\rangle = \lim_{n\longrightarrow\infty} x_n = 0 = \langle\!\langle 0, \beta\rangle\!\rangle \qquad \forall\ \beta=(x_1,x_2,\ldots)\in\ell_2\,,$$

but $\omega_n\not\longrightarrow\omega=0$ since $\|\omega_n-0\|=1$ for all n.

We now verify (2). Since

$$\begin{aligned}\big\|G^2\varphi_n - \eta^2\varphi_n\big\|^2 &= \|G^2\varphi_n\|^2 - 2\eta^2\langle\!\langle G^2\varphi_n, \varphi_n\rangle\!\rangle + \eta^4\|\varphi_n\|^2 \\ &= \|G(G\varphi_n)\|^2 - 2\eta^2\langle\!\langle G^2\varphi_n, \varphi_n\rangle\!\rangle + \eta^4,\end{aligned}$$

it follows that

$$\begin{aligned}\big\|G^2\varphi_n - \eta^2\varphi_n\big\|^2 \le \big(\eta\|G\varphi_n\|\big)^2 - 2\eta^2\langle\!\langle G^2\varphi_n, \varphi_n\rangle\!\rangle + \eta^4 = \eta^2\big(\|G\varphi_n\|-\eta\big)^2 \\ \Longrightarrow \qquad \lim_{n\longrightarrow\infty} \big\|G^2\varphi_n - \eta^2\varphi_n\big\| = 0.\end{aligned}$$

On the other hand, by equation (1) above

$$\begin{aligned}
\eta\big\|G\varphi_n - \eta\varphi_n\big\|^2 &\le \eta\big\langle\!\big\langle G\varphi_n - \eta\varphi_n, G\varphi_n - \eta\varphi_n\big\rangle\!\big\rangle + \big\langle\!\big\langle G\varphi_n - \eta\varphi_n, G(G\varphi_n - \eta\varphi_n)\big\rangle\!\big\rangle \\
&= \big\langle\!\big\langle G\varphi_n - \eta\varphi_n, G^2\varphi_n - \eta^2\varphi_n\big\rangle\!\big\rangle \\
&\Longrightarrow \qquad \lim_{n\longrightarrow\infty}\big\|G\varphi_n - \eta\varphi_n\big\| = 0.
\end{aligned}$$

It follows that for all $\beta \in E^p(M)$,

$$\begin{aligned}
l\big((\Delta - 1/\eta)\beta\big) = \eta\cdot\lim_{n\longrightarrow\infty}\big\langle\!\big\langle G\varphi_n, (\Delta - 1/\eta)\beta\big\rangle\!\big\rangle &= \lim_{n\longrightarrow\infty}\eta\big\langle\!\big\langle (\Delta - 1/\eta)G\varphi_n, \beta\big\rangle\!\big\rangle \\
&= \lim_{n\longrightarrow\infty}\langle\!\langle \eta\varphi_n - G\varphi_n, \beta\rangle\!\rangle = 0,
\end{aligned}$$

since

$$\big|\langle\!\langle \eta\varphi_n - G\varphi_n, \beta\rangle\!\rangle\big| \le \|\beta\|\cdot\|G\varphi_n - \eta\varphi_n\|.$$

(f) Suppose $\lambda \in \mathbb{R}^+$. Let $\mathcal{H}^p_\lambda$ be the subspace of $E^p(M)$ spanned by all eigenfunctions of Δ with eigenvalues less than λ, including 0. Since G is bounded,

$$\eta \equiv \sup_{\varphi\in(\mathcal{H}^p_\lambda)^\perp, \|\varphi\|=1} \|G\varphi\| < \infty.$$

By (b) and (c) above, $\mathcal{H}^p_\lambda$ is a finite-dimensional subspace of the infinite-dimensional vector space $E^p(M)$. Since G is injective on $(\mathcal{H}^p)^\perp$, G is nonzero on $(\mathcal{H}^p_\lambda)^\perp \subset (\mathcal{H}^p)^\perp$ and so $\eta > 0$. It is shown below Δ has an eigenfunction $\omega \in (\mathcal{H}^p_\lambda)^\perp$ with eigenvalue $1/\eta$. Since $\omega \notin \mathcal{H}^p_\lambda$, $1/\eta \ge \lambda$. This implies the claim.

Similarly to part (e), there exists a sequence $\varphi_n \in (\mathcal{H}^p_\lambda)^\perp$ such that

$$\|\varphi_n\| = 1, \qquad \lim_{n\longrightarrow\infty}\|G\varphi_n\| = \eta,$$

and the sequence $\{G\varphi_n\}$ is Cauchy. Define

$$L\colon E^p(M) \longrightarrow \mathbb{R} \qquad \text{by} \qquad L(\beta) = \eta\cdot\lim_{n\longrightarrow\infty}\langle\!\langle G\varphi_n, \beta\rangle\!\rangle \quad \forall\ \beta \in E^p(M).$$

For exactly the same reasons as in part (e), L is a well-defined bounded linear functional such that

$$L\big((\Delta - 1/\eta)^*\beta\big) = \langle\!\langle 0, \beta\rangle\!\rangle \qquad \forall\ \beta \in E^p(M),$$

i.e. L is a weak solution of the equation $(\Delta - 1/\eta)\omega = 0$. As in part (e), we conclude that there exists an eigenfunction $\omega \in E^p(M)$ of Δ with eigenvalue $1/\eta \in \mathbb{R}^+$ such that $L_\omega = \eta\lim_{n\longrightarrow\infty} L_{G\varphi_n}$. By *Remark* in part (e), this implies that $\eta G\varphi_n \longrightarrow \omega$ and so $\omega \in (\mathcal{H}^p_\lambda)^\perp$.

(g) Let $\lambda_1 \le \lambda_2 \le \ldots$ be the eigenvalues of Δ, including 0, listed with multiplicity. Let $u_1, u_2, \ldots$ be an orthonormal set of eigenfunctions of Δ for these eigenvalues. We will show that for every $\alpha \in E^p(M)$,

$$\lim_{n\longrightarrow\infty}\Big\|\alpha - \sum_{i=1}^n \langle\!\langle \alpha, u_i\rangle\!\rangle u_i\Big\| = \lim_{\lambda\longrightarrow\infty}\Big\|\alpha - \sum_{\lambda_i<\lambda}\langle\!\langle \alpha, u_i\rangle\!\rangle u_i\Big\| = 0.$$

The first equality follows from parts (b) and (c).

If $\lambda>0$, the p-form

$$\alpha-\sum_{\lambda_i<\lambda}\langle\!\langle\alpha,u_i\rangle\!\rangle u_i$$

is the orthogonal projection of α onto the subspace $(\mathcal{H}^p_\lambda)^\perp$; see part (f). In particular, it is an element of $(\mathcal{H}^p_\lambda)^\perp$ of norm less than $\|\alpha\|$. Since $(\mathcal{H}^p_\lambda)^\perp\subset(\mathcal{H}^p)^\perp$,

$$\begin{aligned}\alpha-\sum_{\lambda_i<\lambda}\langle\!\langle\alpha,u_i\rangle\!\rangle u_i &= G\Delta\Big(\alpha-\sum_{\lambda_i<\lambda}\langle\!\langle\alpha,u_i\rangle\!\rangle u_i\Big)=G\Big(\Delta\alpha-\sum_{\lambda_i<\lambda}\langle\!\langle\alpha,u_i\rangle\!\rangle\Delta u_i\Big)\\ &= G\Big(\Delta\alpha-\sum_{\lambda_i<\lambda}\langle\!\langle\alpha,u_i\rangle\!\rangle\lambda_i u_i\Big)=G\Big(\Delta\alpha-\sum_{\lambda_i<\lambda}\langle\!\langle\alpha,\lambda_i u_i\rangle\!\rangle u_i\Big)\\ &= G\Big(\Delta\alpha-\sum_{\lambda_i<\lambda}\langle\!\langle\alpha,\Delta u_i\rangle\!\rangle u_i\Big)=G\Big(\Delta\alpha-\sum_{\lambda_i<\lambda}\langle\!\langle\Delta\alpha,u_i\rangle\!\rangle u_i\Big).\end{aligned}$$

Since this is an element of $(\mathcal{H}^p_\lambda)^\perp$, by part (f)

$$\begin{aligned}\Big\|\alpha-\sum_{\lambda_i<\lambda}\langle\!\langle\alpha,u_i\rangle\!\rangle u_i\Big\| &= \Big\|G\Big(\Delta\alpha-\sum_{\lambda_i<\lambda}\langle\!\langle\Delta\alpha,u_i\rangle\!\rangle u_i\Big)\Big\|\le\frac{1}{\lambda}\Big\|\Delta\alpha-\sum_{\lambda_i<\lambda}\langle\!\langle\Delta\alpha,u_i\rangle\!\rangle u_i\Big\|\\ &\le\frac{1}{\lambda}\|\Delta\alpha\|.\end{aligned}$$

This implies the claim.

(h) With notation as in part (g), we will show that

$$\lim_{n\longrightarrow\infty}\Big\|\alpha-\sum_{i=1}^{n}\langle\!\langle\alpha,u_i\rangle\!\rangle u_i\Big\|_\infty=\lim_{\lambda\longrightarrow\infty}\Big\|\alpha-\sum_{\lambda_i<\lambda}\langle\!\langle\alpha,u_i\rangle\!\rangle u_i\Big\|_\infty=0$$

for every $\alpha\in E^p(M)$. By the Sobolev inequality (6.22-(1)), the Fundamental Inequality (6.29-(1)), ellipticity of Δ, and the compactness of M, there exist $C\in\mathbb{R}^+$ and $k\in\mathbb{Z}^+$ such that

$$\|\beta\|_\infty\le C\sum_{m=0}^{m=k}\|\Delta^m\beta\|\qquad\forall\,\beta\in E^p(M).$$

Thus, by the proof of part (g),

$$\begin{aligned}\Big\|\alpha-\sum_{\lambda_i<\lambda}\langle\!\langle\alpha,u_i\rangle\!\rangle u_i\Big\|_\infty &\le C\sum_{m=0}^{m=k}\Big\|\Delta^m\Big(\alpha-\sum_{\lambda_i<\lambda}\langle\!\langle\alpha,u_i\rangle\!\rangle u_i\Big)\Big\|=C\sum_{m=0}^{m=k}\Big\|\Delta^m\alpha-\sum_{\lambda_i<\lambda}\langle\!\langle\Delta^m\alpha,u_i\rangle\!\rangle u_i\Big\|\\ &\le C\sum_{m=0}^{m=k}\frac{1}{\lambda}\|\Delta^{m+1}\alpha\|=\frac{1}{\lambda}\Big(C\sum_{m=0}^{m=k}\|\Delta^{m+1}\alpha\|\Big).\end{aligned}$$

This implies the claim.

Problem 5 (15pts)

Suppose M and N are smooth compact Riemannian manifolds. A differential form γ on $M\times N$ is called decomposable *if*

$$\gamma=\pi_M^*\alpha\wedge\pi_N^*\beta \qquad \text{for some} \quad \alpha\in E^*(M),\ \beta\in E^*(N).$$

(a) *Show that*

$$\Delta_{M\times N}\big(\pi_M^*\alpha\wedge\pi_N^*\beta\big)=\pi_M^*\Delta_M\alpha\wedge\pi_N^*\beta+\pi_M^*\alpha\wedge\pi_N^*\Delta_N\beta \qquad \forall\ \alpha\in E^*(M),\, \beta\in E^*(N).$$

(b) *Show that the $\mathbb{R}$-span of the decomposable forms on $M\times N$ is L^2-dense in $E^*(M\times N)$.*

(c) *Conclude that*

$$\mathcal{H}^*(M\times N)\approx\mathcal{H}^*(M)\otimes\mathcal{H}^*(N), \qquad \pi_M^*\alpha\wedge\pi_N^*\beta\longleftrightarrow\alpha\otimes\beta.$$

(d) *Conclude that*

$$H^p_{\mathrm{de\,R}}(M\times N)\approx\bigoplus_{q+r=p}H^q_{\mathrm{de\,R}}(M)\otimes H^r_{\mathrm{de\,R}}(N), \qquad \pi_M^*\alpha\wedge\pi_N^*\beta\longleftrightarrow\alpha\otimes\beta.$$

This is the Kunneth Formula for de Rham cohomology of compact manifolds.

(a) If X is a smooth manifold of dimension k,

$$\Delta=dd^*+d^*d \qquad \text{and} \qquad d^*\alpha=(-1)^{k(p+1)+1}*d*\alpha \quad \forall\ \alpha\in E^p(X).$$

In the given case, M and N may not oriented, but it is sufficient to check the identity on $U\times V$ for small open subsets U and V of M and N. Let m and n be the dimensions of M and N and suppose $\alpha\in E^p(M)$ and $\beta\in E^q(N)$. On U and V, we can choose orientations, which induce an orientation on $U\times V$. With respect to these orientation,

$$*\big(\pi_M^*\alpha\wedge\pi_N^*\beta\big)=(-1)^{(m-p)q}\pi_M^*(*\alpha)\wedge\pi_N^*(*\beta).$$

Since d commutes with pull-backs, it follows that

$$\begin{aligned}
d^*\big(\pi_M^*\alpha\wedge\pi_N^*\beta\big)&=(-1)^{(m+n)(p+q+1)+1}*d*\big(\pi_M^*\alpha\wedge\pi_N^*\beta\big)\\
&=(-1)^{(m+n)(p+q+1)+1}(-1)^{(m-p)q}*d\big(\pi_M^*(*\alpha)\wedge\pi_N^*(*\beta)\big)\\
&=(-1)^{(m+n)(p+q+1)+1+(m-p)q}*\big(\pi_M^*(d*\alpha)\wedge\pi_N^*(*\beta)+(-1)^{m-p}\pi_M^*(*\alpha)\wedge\pi_N^*(d*\beta)\big)\\
&=(-1)^{(m+n)(p+q+1)+1+(m-p)q}\Big((-1)^{(p-1)(n-q)}(-1)^{q(n-q)}\pi_M^*(*d*\alpha)\wedge\pi_N^*\beta\\
&\qquad+(-1)^{m-p}(-1)^{p(n-q+1)}(-1)^{p(m-p)}\pi_M^*\alpha\wedge\pi_N^*(*d*\beta)\Big)\\
&=(-1)^{m(p+1)+1}\,\pi_M^*(*d*\alpha)\wedge\pi_N^*\beta+(-1)^{n(q+1)+1+p}\pi_M^*\alpha\wedge\pi_N^*(*d*\beta)\\
&=\pi_M^*(d^*\alpha)\wedge\pi_N^*\beta+(-1)^p\pi_M^*\alpha\wedge\pi_N^*(d^*\beta).
\end{aligned}$$

Thus,

$$\begin{aligned}
dd^*\big(\pi_M^*\alpha\wedge\pi_N^*\beta\big)&=d\big(\pi_M^*(d^*\alpha)\wedge\pi_N^*\beta+(-1)^p\pi_M^*\alpha\wedge\pi_N^*(d^*\beta)\big)\\
&=\pi_M^*(dd^*\alpha)\wedge\pi_N^*\beta+(-1)^{p-1}\pi_M^*(d^*\alpha)\wedge\pi_N^*(d\beta)+(-1)^p\pi_M^*d\alpha\wedge\pi_N^*(d^*\beta)+\pi_M^*\alpha\wedge\pi_N^*(dd^*\beta).
\end{aligned}$$

Similarly,

$$\begin{aligned}
d^*d\big(\pi_M^*\alpha\wedge\pi_N^*\beta\big) &= d^*\big(\pi_M^*(d\alpha)\wedge\pi_N^*\beta+(-1)^p\pi_M^*\alpha\wedge\pi_N^*(d\beta)\big)\\
&= \pi_M^*(d^*d\alpha)\wedge\pi_N^*\beta+(-1)^{p+1}\pi_M^*(d\alpha)\wedge\pi_N^*(d^*\beta)+(-1)^p\pi_M^*(d^*\alpha)\wedge\pi_N^*(d\beta)+\pi_M^*\alpha\wedge\pi_N^*(d^*d\beta).
\end{aligned}$$

From the two expressions, we obtain

$$\begin{aligned}
\Delta_{M\times N}\big(\pi_M^*\alpha\wedge\pi_N^*\beta\big) &= \pi_M^*(dd^*\alpha)\wedge\pi_N^*\beta+\pi_M^*\alpha\wedge\pi_N^*(dd^*\beta)+\pi_M^*(d^*d\alpha)\wedge\pi_N^*\beta+\pi_M^*\alpha\wedge\pi_N^*(d^*d\beta)\\
&= \pi_M^*\Delta_M\alpha\wedge\pi_N^*\beta+\pi_M^*\alpha\wedge\pi_N^*\Delta_N\beta,
\end{aligned}$$

as claimed.

(b) Let $\{(U_{M;i},\varphi_{M;i},\psi_{M;i})\}$ and $\{(U_{N;j},\varphi_{N;j},\psi_{N;j})\}$ be finite collections such that

- $\{U_{M;i}\}$ and $\{U_{N;j}\}$ are open covers of M and N, respectively;
- $\varphi_{M;i}\colon U_{M;i}\longrightarrow W_{M;i}\subset\mathbb{R}^m$ and $\varphi_{N;j}\colon U_{N;j}\longrightarrow W_{N;j}\subset\mathbb{R}^n$ are measure-preserving charts;
- $\psi_{M;i}\colon TM|_{U_{M;i}}\longrightarrow W_{M;i}\times\mathbb{R}^m$ and $\psi_{N;j}\colon TN|_{U_{N;j}}\longrightarrow W_{N;j}\times\mathbb{R}^n$ are bundle isometries covering $\varphi_{M;i}$ and $\varphi_{N;j}$, respectively.

Since $M\times N$ is compact, it is sufficient to show that every $\gamma\in E^*(M\times N)$ such that $\operatorname{supp}\gamma\subset U_{M;i}\times U_{N;j}$ for some i,j lies in the closure of the span of decomposable forms. Via the trivializations induced by $\psi_{M;i}$ and $\psi_{N;j}$ on $\Lambda^*(T^*M)$ and $\Lambda^*(T^*N)$, such a form γ corresponds to a smooth compactly supported function on $\mathbb{R}^m\times\mathbb{R}^n$ with values in $\mathbb{R}^p$ for some p. It is sufficient to show that every component function $h=h(x,y)$ can be approximated by a linear combination of functions of the form f_kg_k, where $f_k=f_k(x)$ and $g=g_k(y)$ are compactly supported functions on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively.

It can be assumed that $m,n\ge1$ (otherwise, there is nothing to prove). Let h be as above and $\epsilon>0$. Choose $R>0$ so that $h(x,y)=0$ if $|x_i|>R$ or $|y_j|>R$ for *any* of the components x_i of x or y_j of y (so h is supported inside of the cube $[-R,R]^{m+n}$). Since $[-R,R]^{m+n}$ is compact, there exists $\delta>0$ such that

$$\big|h(x,y)-h(x',y')\big|^2<\frac{\epsilon}{4(2R)^{m+n}}\qquad\text{if}\qquad |x_i-x_i'|,|y_j-y_j'|\le\delta\ \forall\, i,j.$$

It can be assumed that δ divides R (since δ can always be made smaller). Let $\{B_k\}$ be a cover of $[-R,R]^{m+n}$ by some $N>0$ closed cubes with side δ so that $B_k\cap B_{k'}$ is either empty or consists of a face of B_k if $k\neq k'$ (so $\{B_k\}$ breaks $[-R,R]^{m+n}$ into N small cubes). Pick a point $(x_k,y_k)\in B_k$. Let h_k be the function which equals $h(x_k,y_k)$ on B_k and 0 everywhere else. By our assumption on δ,

$$\begin{aligned}
\Big\|h-\sum_{k=1}^{k=N}h_k\Big\|^2=\sum_{k=1}^{k=N}\int_{B_k}\big|h(x,y)-h(x_k,y_k)\big|^2 &< \frac{\epsilon}{4(2R)^{m+n}}\sum_{k=1}^{k=N}\int_{B_k}1\\
&= \frac{\epsilon}{4(2R)^{m+n}}\sum_{k=1}^{k=N}\int_{[-R,R]^{m+n}}1=\frac{\epsilon}{4}\,.
\end{aligned}$$

Let $\delta'\in(0,\delta)$ be such that

$$\max_k|h(x_k,y_k)|^2\cdot\big(\delta^{m+n}-\delta'^{m+n}\big)<\frac{\epsilon}{4N}\,.$$

For each k, $B_k=B_{m;k}\times B_{n;k}$ for some cubes $B_{m;k}\subset\mathbb{R}^m$ and $B_{n;k}\subset\mathbb{R}^n$ of side δ. Let $B'_{m;k}\subset\operatorname{Int}B_{m;k}$ and $B'_{n;k}\subset\operatorname{Int}B_{n;k}$ be closed cubes with side δ'. For each k, choose smooth functions

$$f_k\colon\mathbb{R}^m\longrightarrow[0,1]\quad\text{and}\quad g'_k\colon\mathbb{R}^n\longrightarrow[0,1]\qquad\text{s.t.}$$
$$\operatorname{supp}f_k\subset B_{m;k},\ \operatorname{supp}g'_k\subset B_{n;k},\ f_k|_{B'_{m;k}}\equiv1,\ g'_k|_{B'_{n;k}}\equiv1.$$

Let $g_k=h(x_k,y_k)g'_k$. Then, for every k

$$\big\|h_k-f_kg_k\big\|^2=\int_{B_k-B'_{m;k}\times B'_{n;k}}\big|h(x_k,y_k)-f_kg_k\big|^2\le\big|h(x_k,y_k)\big|^2\cdot\big(\delta^{m+n}-\delta'^{m+n}\big)<\frac{\epsilon}{4N}.$$

Putting this all together, we obtain

$$\Big\|h-\sum_{k=1}^{k=N}f_kg_k\Big\|^2\le2\bigg(\Big\|h-\sum_{k=1}^{k=N}h_k\Big\|^2+\sum_{k=1}^{k=N}\big\|h_k-f_kg_k\big\|^2\bigg)<2\bigg(\frac{\epsilon}{4}+\frac{\epsilon}{4N}\cdot N\bigg)=\epsilon,$$

so h is in the closure of the span of decomposable elements f_kg_k.

Remark: The argument suggested in Griffiths&Harris, Lemma on p104, is wrong. If A is a subspace of a Hilbert space H (like L^2-forms), then $\bar{A}=H$ if and only if for every $h\in H-0$ there exists $f\in A$ such that $\langle\langle f,h\rangle\rangle\neq0$. The *only if* part is immediate and does not depend on the completeness of H (if $h\in H-0$ and $\langle\langle f,h\rangle\rangle=0$ for all $f\in A$, then $\|h-f\|\ge\|h\|$ for every $f\in A$ and so $h\notin\bar{A}$). On the other hand, $H=\bar{A}\oplus\bar{A}^\perp$ because H is a Hilbert space and $\bar{A}\subset H$ is closed; thus, if $\bar{A}\neq H$, then there exists $h\in H-0$ such that $\langle\langle f,h\rangle\rangle=0$ for all $f\in A$. This last implication does not need to hold if H is an inner-product space which is not complete. For example, let

$$H=C^\infty(I;\mathbb{R}),\qquad A=\Big\{h\in H\colon\int_0^{1/2}h\mathrm{d}x=0\Big\},$$

where $I=[0,1]$ and H has the L^2-norm. Since

$$H\longrightarrow\mathbb{R},\qquad h\longrightarrow\int_0^{1/2}h\mathrm{d}x\,,$$

is a bounded linear functional on H (L^1-norm is bounded by L^2-norm), $\bar{A}=A\neq H$. On the other hand, for every $h\in H-0$, there exists $f\in A$ such that $\langle\langle f,h\rangle\rangle\neq0$. If $h(x)\neq0$ for some $x\in[1/2,1]$, such an f can be constructed using a cut-off function supported on a small neighborhood of some $x_0\in(1/2,1)$. Otherwise, we can assume that there exists $x_0\in(0,1/2)$ such that $h(x_0)>0$ (after possibly replacing h by $-h$) and $h'(x_0)<0$ (because $h(1/2)=0$). Thus, there exists $\delta>0$ such that

$$(x_0-\delta,x_0+\delta)\subset(0,1/2)\quad\text{and}\quad h(x)>h(x_0)\ \forall\,x\in(x_0-\delta,x_0),\quad h(x)<h(x_0)\ \forall\,x\in(x_0,x_0+\delta).\tag{4}$$

Let $f\in C^\infty(\mathbb{R};\mathbb{R})$ be any nonzero function such that

$$\operatorname{supp}f\subset(x_0-\delta,x_0+\delta)\quad\text{and}\quad f(x_0+y)=-f(x_0-y)\ \ \forall\,y\in\mathbb{R},\ \ f(x)\le0\ \ \forall\,x\in(x_0,x_0+\delta).\tag{5}$$

Thus, $f|_I\in A$ and

$$\langle\langle f|_I,h\rangle\rangle=\int_{x_0-\delta}^{x_0}f\cdot(h-h(x_0))\,\mathrm{d}x+\int_{x_0}^{x_0+\delta}f\cdot(h-h(x_0))\,\mathrm{d}x+h(x_0)\int_0^1f\mathrm{d}x>0,$$

because the integrands in the first two integrals are non-negative and positive somewhere by (4) and (5) and $f\in A$. A quicker way to see that the orthogonal complement of A in H is zero is to pass to $L^2(I;\mathbb{R})\supset H$. Since

$$L^2(I;\mathbb{R}) \longrightarrow \mathbb{R}, \qquad h \longrightarrow \int_0^{1/2} h\mathrm{d}x\,,$$

is a well-defined bounded linear surjective functional, the orthogonal complement of its kernel is one-dimensional (the kernel is the closure of A in $L^2(I;\mathbb{R})$). The orthogonal complement of A in $L^2(I;\mathbb{R})$ thus consists of the functions $h\colon I\longrightarrow\mathbb{R}$ that are constant on $[0,1/2]$ and vanish on $(1/2,1]$ (because these functions are indeed orthogonal to A). Since the only one of these functions that lies in H is the zero function, the orthogonal complement of A in H is zero.

(c) Let $\lambda_1\le\lambda_2\le\ldots$ and $\tau_1\le\tau_2\le\ldots$ be the eigenvalues of Δ_M and Δ_N, including zero and listed with multiplicity. Let

$$\alpha_1,\alpha_2,\ldots\in E^*(M) \qquad \text{and} \qquad \beta_1,\beta_2,\ldots\in E^*(N)$$

be orthonormal eigenfunctions for these eigenvalues. By part (g) of Problem 4, their linear spans are L^2-dense in $E^*(M)$ and in $E^*(N)$. Thus, the linear span of the vectors $\pi_M^*\alpha_i\wedge\pi_N^*\beta_j$ is dense in the span of the decomposable elements of $E^*(M\times N)$. Since this span is L^2-dense in $E^*(M\times N)$ by part (b), the linear span of the vectors $\pi_M^*\alpha_i\wedge\pi_N^*\beta_j$ is L^2-dense in $E^*(M\times N)$. On the other hand, by part (a),

$$\begin{aligned}\Delta_{M\times N}\big(\pi_M^*\alpha_i\wedge\pi_N^*\beta_j\big) &= \pi_M^*(\Delta_M\alpha_i)\wedge\pi_N^*\beta_j+\pi_M^*\alpha_i\wedge\pi_N^*(\Delta_N\beta_j)\\ &= \pi_M^*(\lambda_i\alpha_i)\wedge\pi_N^*\beta_j+\pi_M^*\alpha_i\wedge\pi_N^*(\tau_j\beta_j)\\ &= (\lambda_i+\tau_j)\cdot\big(\pi_M^*\alpha_i\wedge\pi_N^*\beta_j\big),\end{aligned}$$

i.e. $\pi_M^*\alpha_i\wedge\pi_N^*\beta_j$ is an eigenfunction for $\Delta_{M\times N}$ with eigenvalue $\lambda_i+\tau_j$. Since the sequences

$$0\le\lambda_1\le\lambda_2\le\ldots \qquad \text{and} \qquad 0\le\tau_1\le\tau_2\le\ldots$$

have no limit points by (b) and (c) of Problem 4, neither does the set $\{\lambda_i+\tau_j : i,j\ge1\}$. Since the linear span of the vectors $\pi_M^*\alpha_i\wedge\pi_N^*\beta_j$ is L^2-dense in $E^*(M\times N)$, it follows that $\Delta_{M\times N}$ has no other eigenvalues and all its eigenvectors are linear combinations of the forms $\pi_M^*\alpha_i\wedge\pi_N^*\beta_j$ with the same value of $\lambda_i+\tau_j$. Since $\lambda_i,\tau_j\ge0$, the zero eigenspace of $\Delta_{M\times N}$ is thus given by

$$\mathcal{H}^*_{M\times N}=\mathrm{Span}\big(\{\pi_M^*\alpha\wedge\pi_N^*\beta\colon \alpha\in\mathcal{H}^*_M,\ \beta\in\mathcal{H}^*_N\}\big)\approx\mathcal{H}^*_M\otimes\mathcal{H}^*_N.$$

(d) The diagram of graded vector-space homomorphisms[1]

$$\begin{tikzcd}
\mathcal{H}^*_M\otimes\mathcal{H}^*_N \arrow[r] \arrow[d] & H^*_{\mathrm{de\,R}}(M)\otimes H^*_{\mathrm{de\,R}}(N) \arrow[d] \\
\mathcal{H}^*_{M\times N} \arrow[r] & H^*_{\mathrm{de\,R}}(M\times N)
\end{tikzcd}
\qquad
\begin{tikzcd}
\alpha\otimes\beta \arrow[r] \arrow[d] & {[\alpha]\otimes[\beta]} \arrow[d] \\
\pi_M^*\alpha\wedge\pi_N^*\beta \arrow[r] & \big[\pi_M^*\alpha\wedge\pi_N^*\beta\big]
\end{tikzcd}$$

commutes. By part (c), the left arrow in the diagram is an isomorphism. By Theorem 6.11, the horizontal arrows are isomorphisms. Thus, so is the right arrow. Restricting to the p-th level, we obtain the desired statement.

[1]these are actually algebra homomorphisms with respect to $\wedge$