Part I (choose 2 problems from 1, 2, and 3)

1. Let \( f: \mathbb{R}P^3 \rightarrow T^3 \equiv (S^1)^3 \) be a smooth map. Show that \( f \) is not an immersion.

   Suppose \( f \) is an immersion. Since \( \mathbb{R}P^3 \) and \( T^3 \) have the same dimension, the differential
   \[
d_x f: T_x \mathbb{R}P^3 \rightarrow T_{f(x)} T^3
   \]
   is an isomorphism for every \( x \in \mathbb{R}P^3 \). By the Inverse Function Theorem, \( f \) is thus a local diffeomorphism, and so its image is open in \( T^3 \). Since \( \mathbb{R}P^3 \) is compact and \( T^3 \) is Hausdorff, \( f(\mathbb{R}P^3) \) is closed in \( T^3 \). Since \( T^3 \) is connected, it follows that \( f \) is surjective. Since \( \mathbb{R}P^3 \) is compact and \( f \) is a local diffeomorphism, \( f^{-1}(y) \subset \mathbb{R}P^3 \) is finite for every \( y \in T^3 \). Thus, \( f \) is a covering projection (the intersection of the images of neighborhoods of elements of \( f^{-1}(y) \) on which \( f \) is a diffeomorphism is an evenly covered neighborhood of \( y \)), and
   \[
f_*: \pi_1(\mathbb{R}P^3, x_0) \rightarrow \pi_1(T^3, f(x_0))
   \]
   is an injective homomorphism. However, this is impossible, since \( \pi_1(\mathbb{R}P^3, x_0) \approx \mathbb{Z}_2 \) has torsion, while \( \pi_1(T^3, f(x_0)) \approx \mathbb{Z}^3 \) is torsion-free.

2. Let \( X \) and \( Y \) be the vector fields on \( \mathbb{R}^3 \) given by
   \[
   X = \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}, \quad Y = y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.
   \]

   (a) Compute the flows \( \varphi_s \) and \( \psi_t \) of \( X \) and \( Y \) (give formulas).

   (b) Do these flows commute?

   (a) The time \( s \)-flow of \( X \) through \((x_0, y_0, z_0)\) is the solution to the initial-value problem
   \[
   \begin{cases}
   x'(s) = 1, & y'(s) = x, \quad z'(s) = y, \\
   (x(0), y(0), z(0)) = (x_0, y_0, z_0).
   \end{cases}
   \]

   Solving the first equation, then the second, and finally the third, we find that
   \[
   (x(s), y(s), z(s)) = (x_0 + s, y_0 + x_0 s + \frac{s^2}{2}, z_0 + y_0 s + x_0 s^2 + \frac{s^3}{6}).
   \]

   Thus, the time \( s \)-flow of \( X \) is given by
   \[
   \varphi_s(x, y, z) = (x + s, y + sx + \frac{s^2}{2}, z + sy + \frac{s^2}{2}x + \frac{s^3}{6}).
   \]

   Similarly, the time \( t \)-flow of \( Y \) is given by
   \[
   \psi_t(x, y, z) = (x + ty + \frac{t^2}{2}z + \frac{t^3}{6}, y + tz + \frac{t^2}{2}, z + t),
   \]
as the roles of $x$ and $z$ in $X$ and $Y$ are interchanged.

(b) Since the Lie bracket of coordinate vector fields is 0,

$$[X,Y] = \left( X(y) \frac{\partial}{\partial x} + X(z) \frac{\partial}{\partial y} + X(1) \frac{\partial}{\partial z} \right) - \left( Y(1) \frac{\partial}{\partial x} + Y(x) \frac{\partial}{\partial y} + Y(y) \frac{\partial}{\partial z} \right) = \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) - \left( 0 + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) = x \frac{\partial}{\partial x} - z \frac{\partial}{\partial z}.$$

Since $[X,Y] \neq 0$, the flows of $X$ and $Y$ do not commute by PS4 #5.

Alternatively, the first coordinates of $\varphi_s \circ \psi_t$ and $\psi_t \circ \varphi_s$ are given by

$$(x, y, z) \mapsto x + ty + \frac{t^2}{2} z + \frac{t^3}{6} + s, (x+s) + t(y+sx + \frac{s^2}{2}) + \frac{t^2}{2} (z+sy + \frac{s^2}{2} x + \frac{s^3}{6}) + \frac{t^3}{6},$$

respectively. Since these are not the same (unless $s=0$ or $t=0$), the flows do not commute.

3. Let $M$ and $N$ be smooth oriented connected manifolds and $H : M \times [0,1] \rightarrow N$ a smooth map. For each $t \in [0,1]$, define

$$H_t : M \rightarrow N, \quad H_t(p) = H(p,t).$$

(a) Suppose $H_t$ is a diffeomorphism for every $t \in [0,1]$. Show that $H_0$ is orientation-preserving if and only if $H_1$ is.

(b) Suppose instead that $M$ is compact and $H_0, H_1$ are diffeomorphisms. Show that $H_0$ is orientation-preserving if and only if $H_1$ is.

(c) Give an example so that $H_0$ and $H_1$ are diffeomorphisms, with $H_0$ orientation-preserving and $H_1$ orientation-reversing.

It can be assumed that the manifolds $M$ and $N$ have the same dimension $n$. Let $\omega_M \in E^n(M)$ and $\omega_N \in E^n(N)$ be oriented volume forms (nowhere 0 top forms). Let $f \in C^\infty(M \times [0,1])$ and $\gamma \in E^{n-1}(M \times [0,1])$ be such that

$$H^* \omega_N = f \cdot \pi_1^* \omega_M + \gamma \wedge \pi_2^* dt \quad \Longrightarrow \quad H_t^* \omega_N = f_t \omega_M,$$

where $f_t \in C^\infty(M)$, $f_t(p) = f(p,t)$.

(a) Since $H_t$ is a diffeomorphism for all $t$, $f(t,p) = f_t(p) \in \mathbb{R}^n$. Since $M \times [0,1]$ is connected, either $f(t,p) \in \mathbb{R}^+$ for all $(t,p)$ or $f(t,p) \in \mathbb{R}^-$ for all $(t,p)$. Thus, $H_0$ is orientation-preserving (i.e. $f_0(p) > 0$ for all $p \in M$) if and only if $H_1$ is (i.e. $f_1(p) > 0$ for all $p \in M$).

(b) Since the maps $H_0, H_1 : M \rightarrow N$ are smoothly homotopic,

$$[H_0^* \omega_N] = [H_1^* \omega_N] \quad \Longrightarrow \quad \int_M H_0^* \omega_N = \int_M H_1^* \omega_N.$$

Since $M$ is connected, either $f_0(p) > 0$ for all $p \in M$ or $f_0(p) < 0$ for all $p \in M$; in the first case

$$\int_M H_0^* \omega_N = \int_M f_0 \omega_M > 0.$$
while in the second case this integral is negative. The same applies to \( f_1 \) and \( H_1 \). Since the two integrals are the same, \( H_0 \) is orientation-preserving (i.e. \( f_0(p) > 0 \) for all \( p \in M \)) if and only if \( H_1 \) is (i.e. \( f_1(p) > 0 \) for all \( p \in M \)).

(c) Let \( H : \mathbb{R} \times [0, 1] \to \mathbb{R} \) be given by

\[
H(p, t) = -tp + (1 - t)p.
\]

Then, \( H_0 = \text{id}_\mathbb{R} \) is orientation-preserving, while \( H_1 = -\text{id}_\mathbb{R} \) is orientation-reversing.

Note: In this case, \( M = \mathbb{R} \) is not compact and \( H_{1/2} \) is the constant map sending \( \mathbb{R} \) to 0 and so is not a diffeomorphism.

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**Part II** (choose 2 problems from 4, 5, and 6)

4. Let \( M \) be a smooth manifold obtained by identifying two copies of a Mobius Band, \( M_1 \) and \( M_2 \), along their boundary circles. Compute \( H^*_\text{deR}(M) \).

Since \( M \) is 2-manifold, \( H^k_\text{deR}(M) = 0 \) for \( k \neq 0, 1, 2 \). Since \( M \) is connected, \( H^0_\text{deR}(M) \approx \mathbb{R} \). Since the interior of \( M_1 \) is a non-orientable open subset of \( M \), \( M \) is also non-orientable, and so \( H^2_\text{deR}(M) \approx 0 \). It remains to compute \( H^1_\text{deR}(M) \).

This can be done using Mayer-Vietoris. Let \( U \subset M \) be a small tubular neighborhood of \( M_1 \) (or the complement of the “equator” in \( M_2 \)) and \( V \subset M \) a small tubular neighborhood of \( M_2 \) (or the complement of the “equator” in \( M_1 \)). Thus, \( U \) and \( V \) are open Mobius Bands, while \( U \cap V \) is an open cylinder. Since all three are homotopic to a circle, by the homotopy invariance of de Rham cohomology,

\[
H^k_\text{deR}(U), H^k_\text{deR}(V), H^k_\text{deR}(U \cap V) \approx H^k_\text{deR}(S^1) \approx \begin{cases} \mathbb{R}, & \text{if } k = 0, 1; \\ 0, & \text{otherwise}. \end{cases}
\]

The MV long sequence in this case is

\[
0 \to H^0_\text{deR}(M) \to H^0_\text{deR}(U) \oplus H^0_\text{deR}(V) \to H^0_\text{deR}(U \cap V) \xrightarrow{\delta_0} H^1_\text{deR}(M) \to H^1_\text{deR}(U) \oplus H^1_\text{deR}(V) \to H^1_\text{deR}(U \cap V) \to H^2_\text{deR}(M).
\]

Plugging in for the known groups, we obtain

\[
0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \xrightarrow{\delta_0} H^1_\text{deR}(M) \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \to 0.
\]

Thus, \( \delta_0 \) is the zero homomorphism, and the sequence

\[
0 \to H^1_\text{deR}(M) \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \to 0
\]

is exact. So, \( H^1_\text{deR}(M) \approx \mathbb{R} \).

Alternatively, the manifold \( M \) is homeomorphic to the Klein bottle, as can be seen from the following diagram (see Chapter 8 in Munkres):

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3
By the last diagram, Hurewicz Theorem, Universal Coefficient Theorem, and de Rham Theorem,
\[
\pi_1(M) = \langle a, b|abab^{-1}\rangle \quad \implies \quad H_1(M;\mathbb{Z}) \approx \text{Abel}(\pi_1(M)) \approx \mathbb{Z}_2 \oplus \mathbb{Z}
\]
\[
\implies \quad H_1(M;\mathbb{R}) \approx H_1(M;\mathbb{Z}) \otimes \mathbb{R} \approx \mathbb{R} \quad \implies \quad H_{deR}^1(M) \approx H_1(M;\mathbb{R})^* \approx \mathbb{R}.
\]

5. Let $M$ be a smooth manifold admitting an open cover \(\{U_i\}_{i=1,...,m}\) such that every intersection $U_1 \cap \ldots \cap U_{i_k}$ is either empty or diffeomorphic to $\mathbb{R}^n$. Show that

(a) if $m=2$, $H_{deR}^p(M) = 0$ for all $p \neq 0$;

(b) if $m \geq 2$, $H_{deR}^p(M) = 0$ for all $p \geq m-1$.

(a) By Mayer-Vietoris, there is a long exact sequence

\[
0 \to H_{deR}^0(M) \to H_{deR}^0(U_1) \oplus H_{deR}^0(U_2) \to H_{deR}^0(U_1 \cap U_2) \to \cdots
\]

Since $H_{deR}^p(U_1 \cap U_2), H_{deR}^{p+1}(U_1), H_{deR}^{p+1}(U_2) = 0$ for all $p \geq 1$, $H_{deR}^{p+1}(M) = 0$ for all $p \geq 1$. If $U_1 \cap U_2 = \emptyset$, then this statement applies for $p=0$ as well. If $U_1 \cap U_2 \neq \emptyset$, $M$ is connected, and the sequence

\[
0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \xrightarrow{\delta_0} H_{deR}^1(M) \to 0
\]
is exact. So, $\delta_0 = 0$ and $H_{deR}^1(M) = 0$.

(b) Suppose the statement holds for some $m \geq 2$ (part (a) is the $m=2$ case); we use Mayer-Vietoris to show that it holds for $m+1$. Let

\[
U = U_1 \cup U_2 \cup \ldots \cup U_m, \quad V = U_{m+1}.
\]

By MV, the sequence

\[
H_{deR}^p(U \cap V) \xrightarrow{\delta_p} H_{deR}^{p+1}(M) \to H_{deR}^{p+1}(U) \oplus H_{deR}^{p+1}(V)
\]
is exact. By the inductive assumption, $H_{deR}^p(U), H_{deR}^p(U \cap V) = 0$ for all $p \geq m-1$. Since $H_{deR}^p(V) = 0$ for $p \geq 1$, the outer terms of the above exact sequence vanish if $p \geq m-1$. Thus, $H_{deR}^{p+1}(M) = 0$ if $p+1 \geq (m+1)-1$, as needed for the inductive step.
6. (a) Explain why \(\mathbb{RP}^2 \times \mathbb{RP}^4\) is not orientable.
(b) Describe the orientable double cover \(M\) of \(\mathbb{RP}^2 \times \mathbb{RP}^4\).
(c) Determine the de Rham cohomology of \(M\).

(a) If \(M\) and \(N\) are smooth nonempty manifolds, \(M \times N\) is orientable if and only if \(M\) and \(N\) are orientable; see MT06 \#5. The even-dimensional projective spaces, \(\mathbb{RP}^2\) and \(\mathbb{RP}^4\), are not orientable.

(b) The universal cover of \(\mathbb{RP}^2 \times \mathbb{RP}^4\) is \(\tilde{M} = S^2 \times S^4\) (because the latter is connected and simply connected and admits a covering map to the former). The group of deck transformations is

\[ G = \{\text{id} \times \text{id}, a_1 \times \text{id}, \text{id} \times a_2, a_1 \times a_2\} \approx \pi_1(M) = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \]

where \(a_1 : S^2 \rightarrow S^2\) and \(a_2 : S^4 \rightarrow S^4\) are the antipodal maps. The orientable double cover \(M\) is the quotient of \(\tilde{M}\) by a subgroup of \(G\) of index 2 and thus of order 2. There are three such subgroups. The quotients of \(\tilde{M}\) by \(\{\text{id} \times \text{id}, a_1 \times \text{id}\}\) and \(\{\text{id} \times \text{id}, \text{id} \times a_2\}\) are \(\mathbb{RP}^2 \times S^4\) and \(S^2 \times \mathbb{RP}^4\); these are non-orientable manifolds, since one of the components in each product is non-orientable. Thus, \(M = \tilde{M} / \{\text{id} \times \text{id}, a_1 \times a_2\} \equiv S^2 \times S^4\), \((x, y) \sim (-x, -y)\).

(c) Since \(M\) is a 6-manifold, \(H^k_{\text{deR}}(M) = 0\) unless \(0 \leq k \leq 6\). Since \(M\) is compact connected and orientable, \(H^0_{\text{deR}}(M), H^6_{\text{deR}}(M) \approx \mathbb{R}\). By PS7 \#5,

\[ H^k_{\text{deR}}(M) \approx H^k_{\text{deR}}(\tilde{M})^\mathbb{Z}_2 \equiv \{[\tilde{\alpha}] \in H^k_{\text{deR}}(\tilde{M}) : \{a_1 \times a_2\}^*[\tilde{\alpha}] = [\tilde{\alpha}]\}. \]

By Kunneth’s formula, the isomorphism

\[ \bigoplus_{p+q=k} H^p_{\text{deR}}(S^2) \otimes H^q_{\text{deR}}(S^4) \longrightarrow H^k_{\text{deR}}(S^2 \times S^4), \quad \beta \otimes \gamma \longrightarrow \{\pi_1^* \beta \wedge \pi_2^* \gamma\}, \]

is an isomorphism. In particular,

\[ H^1_{\text{deR}}(\tilde{M}), H^3_{\text{deR}}(\tilde{M}), H^5_{\text{deR}}(M) = 0 \quad \Rightarrow \quad H^1_{\text{deR}}(M), H^3_{\text{deR}}(M), H^5_{\text{deR}}(M) = 0. \]

On the other hand, let \([\omega_1]\) and \([\omega_2]\) be the generators of \(H^2_{\text{deR}}(S^2) \approx \mathbb{R}\) and \(H^4_{\text{deR}}(S^4) \approx \mathbb{R}\), respectively. By the solution to PS6 \#6a, \(a_1^*[\omega_1] = (-1)^{2+1}[\omega_1]\) and so

\[ \{a_1 \times a_2\}^*\pi_1^*[\omega_1] = \{\pi_1 \circ a_1 \times a_2\}^*[\omega_1] = \{a_1 \circ \pi_1\}^*[\omega_1] = \pi_1^*a_1^*[\omega_1] = -\pi_1^*[\omega_1]. \]

Similarly, \(\{a_1 \times a_2\}^*\pi_2^*[\omega_2] = -\pi_2^*[\omega_2]\). Thus,

\[ H^2_{\text{deR}}(M) \approx H^2_{\text{deR}}(\tilde{M})^\mathbb{Z}_2 = 0, \quad H^4_{\text{deR}}(M) \approx H^4_{\text{deR}}(\tilde{M})^\mathbb{Z}_2 = 0. \]
Part III (choose 1 problem from 7 and 8)

7. Let $V, W \to S^1$ be smooth real vector bundles. Show that at least one of the vector bundles $V, W, V \oplus W \to S^1$ is orientable.

This is equivalent to showing that at least one of the real line bundles

$$\Lambda^\text{top} V, \Lambda^\text{top} W, \Lambda^\text{top} (V \oplus W) \approx \Lambda^\text{top} V \otimes \Lambda^\text{top} W \to S^1$$

is trivial; see Lemma 15.1 in Lecture Notes. The set of isomorphism classes of real line bundles with the tensor product is an abelian group isomorphic to $\check{H}^1(S^1; \mathbb{Z}_2)$, with the trivial line bundle corresponding to $0 \in \check{H}^1(S^1; \mathbb{Z}_2)$; see PS9 #2a. The latter group is isomorphic to

$$H_1(S^1; \mathbb{Z}) \approx \text{Abel}(\pi_1(S^1)).$$

Thus, for any $v, w \in \check{H}^1(S^1; \mathbb{Z}_2)$, at least one of the three elements $v, w, v+w$ is zero.

Here is a direct argument. Let $\{U_i\}_{i=1,2,\ldots,k}$, with $k \geq 4$, be a cover of $S^1$ by open intervals such that $U_i \cap U_j = \emptyset$ unless $i = j$ or $i \equiv j \pm 1 \mod n$, trivalizations of $V$ and $W$, and

$$g_{i,j}^V : U_i \cap U_j \to \text{GL}_l \mathbb{R} \quad \text{and} \quad g_{i,j}^W : U_i \cap U_j \to \text{GL}_m \mathbb{R}$$

the corresponding transition data. The maps

$$g_{i,j}^{V \oplus W} = g_{i,j}^V \oplus g_{i,j}^W : U_i \cap U_j \to \text{GL}_{l+m} \mathbb{R}$$

are then transition data for $V \oplus W$. By our assumptions, $U_i \cap U_j$ is a connected interval and thus $\det g_{i,j}$ does not change sign on $U_i \cap U_j$. By negating the first component of $h_{i+1}^V$ and $h_{i+1}^W$ if necessary, we can assume that

$$\det g_{i,i+1}^V, \det g_{i,i+1}^W > 0 \quad \forall i = 1, 2, \ldots, n-1.$$

If $\det g_{n,1}^V > 0$, then $V$ is orientable; see Lemma 15.1 in Lecture Notes. If $\det g_{n,1}^V, \det g_{n,1}^W < 0$, then

$$\det g_{i,j}^{V \oplus W} = \det g_{i,j}^V \cdot \det g_{i,j}^W > 0 \quad \forall i, j = 1, 2, \ldots, n.$$

So, if $V, W \to S^1$ are not orientable, then $V \oplus W \to S^1$ is orientable.
8. Let \( \pi: V \rightarrow M \) be a smooth vector bundle. A connection in \( V \) is an \( \mathbb{R} \)-linear map

\[
\nabla : \Gamma(M; V) \rightarrow \Gamma(M; T^*M \otimes V) \quad \text{s.t.} \quad \nabla(fs) = df \otimes s + f \nabla s \quad \forall f \in C^\infty(M), \ s \in \Gamma(M; V).
\]

(a) Show that \( \nabla \) is a first-order differential operator.
(b) What is the symbol of \( \nabla ? \)
(c) Under what conditions (on \( M \) and/or \( V \)) is \( \nabla \) elliptic?

(a) First, \( \nabla \) is a local operator, i.e. the value of \( \nabla s \) at a point \( p \in M \) depends only on the restriction of \( s \) to any neighborhood \( U \) of \( p \). If \( f \) is a smooth function on \( M \) supported in \( U \) such that \( f(p) = 1 \), then

\[
\nabla s|_p = \nabla (fs)|_p - d_pf \otimes s(p),
\]

by the product-rule condition. The right-hand side of this expression depends only on \( s|_U \).

Let \( \varphi \equiv (x_1, \ldots, x_n) : U \rightarrow \mathbb{R}^n \) be a chart on \( M \). An isomorphism \( \tilde{\Psi} : V|_U \rightarrow \mathbb{R}^n \times \mathbb{R}^k \) of vector bundles covering \( \varphi \) induces such an isomorphism for the bundle \( T^*M \otimes V \):

\[
\tilde{\Psi} : T^*M \otimes V|_U \rightarrow \mathbb{R}^n \times (\mathbb{R}^k)^n,
\]

\[
\eta \mapsto \left( p, \eta \left( \frac{\partial}{\partial x_1}|_p \right), \ldots, \eta \left( \frac{\partial}{\partial x_n}|_p \right) \right) \quad \forall \eta \in T^*_p M \otimes V_p, \ p \in U.
\]

For each \( i=1,2,\ldots, k \), define

\[
s_i \in \Gamma(U; V) \quad \text{by} \quad s_i(p) = \psi^{-1}(\varphi_i(p), e_i) \quad \forall p \in U,
\]

where \( e_i \in \mathbb{R}^k \) is the \( i \)-th standard coordinate vector. The homomorphisms

\[
\tilde{\psi} : C^\infty(\mathbb{R}^n; \mathbb{R}^k) \rightarrow \Gamma(U; V)
\]

\[
\{ \tilde{\psi}(f_1, \ldots, f_k) \}(p) = \sum_{i=1}^{i=k} f_i(\phi(p)) s_i(p),
\]

\[
\tilde{\Phi} : C^\infty(\mathbb{R}^n; (\mathbb{R}^k)^n) \rightarrow \Gamma(U; T^*M \otimes V)
\]

\[
\{ \tilde{\Phi}(f_{j,l})_{j=1,\ldots,n; l=1,2,\ldots,k} \}(p) = \sum_{j=1}^{j=n} \sum_{l=1}^{l=k} f_{j,l}(\phi(p)) d_p x_j \otimes s_l(p),
\]

are then isomorphisms. By definition of \( \nabla \), there exist

\[
\theta_{j,l}^i \in C^\infty(U) \quad \text{s.t.} \quad \nabla s_i|_p = \sum_{j=1}^{j=n} \sum_{l=1}^{l=k} \theta_{j,l}^i(p) d_p x_j \otimes s_l(p) \quad \forall p \in U.
\]

By the product-rule condition on \( \nabla \),

\[
\nabla (\tilde{\psi}(f_1, \ldots, f_k))|_p = \sum_{i=1}^{i=k} d_p (f_i \circ \phi) \otimes s_i(p) + \sum_{i=1}^{i=k} \sum_{j=1}^{j=n} \sum_{l=1}^{l=k} \theta_{j,l}^i(p) f_i(\phi(p)) d_p x_j \otimes s_l(p)
\]

\[
= \sum_{j=1}^{j=n} \sum_{l=1}^{l=k} \left( \frac{\partial (f_i \circ \phi)}{\partial x_j} \right)_p + \sum_{i=1}^{i=k} \theta_{j,l}^i(p) f_i(\phi(p)) \right) d_p x_j \otimes s_l(p).
\]
Thus, the operator $\nabla|_U$ in the local coordinates $(\varphi, \psi, \Psi)$ on $(U, V|_U, T^*M \otimes V|_U)$ is given by

$$\tilde{\Psi}^{-1} \circ \nabla \circ \tilde{\psi} : C^\infty(\mathbb{R}^n; \mathbb{R}^k) \to C^\infty(\mathbb{R}^n; (\mathbb{R}^k)^n),$$

$$(f_i)_{i=1,2,...,k} \to \left( \frac{\partial f_i}{\partial x_j} + \sum_{i=1}^{i=k} \theta_{j,d}^i \circ \varphi^{-1} \cdot f_i \right)_{j=1,2,...,n; \ell=1,2,...,k}.$$

Since this is a first-order differential operator on functions on $\mathbb{R}^n$, $\nabla$ is a first-order differential operator on vector-bundle sections over $M$.

(b) Let $p \in M$, $\alpha \in T^*_p M$, $f \in C^\infty(M)$ be such that $f(p) = 0$ and $d_p f = \alpha$, and $s \in \Gamma(M; V)$. By the product-rule condition on $\nabla$,

$$\nabla(fs)|_p = d_p f \otimes s + f(p) \otimes (\nabla s)|_p = \alpha \otimes s(p).$$

Thus, the symbol of $\nabla$ is given by

$$\sigma_\nabla : T^*M \to \text{Hom}(V, T^*M \otimes V), \quad \{\sigma_\nabla(\alpha)\}(v) = \alpha \otimes v \quad \forall \alpha \in T^*_p M, \ v \in V_p, \ p \in M.$$

(c) The operator $\nabla$ is elliptic if and only if the homomorphism

$$\sigma_\nabla(\alpha) : V_p \to T^*_p M \otimes V_p$$

is an isomorphism for all $\alpha \in T^*_p M - 0$ and $p \in M$. If this is the case (and $V$ has positive rank), then

$$\text{rk} V = \text{rk} T^*M \otimes V \quad \implies \quad \dim M = 1.$$

Conversely, if $\dim M = 1$, $\sigma_\nabla(\alpha)$ is an isomorphism for all $\alpha \in T^*_p M - 0$ and $p \in M$. Thus, $\nabla$ is elliptic if and only if $\dim M = 1$ (or $\text{rk} V = 0$).

**Bonus Problem**

Let $\gamma \to \mathbb{C}P^1$ be the tautological (complex) line bundle. Compute

$$\int_{\mathbb{C}P^1} c_1(\gamma^*),$$

where $\mathbb{C}P^1$ has its canonical orientation as a complex manifold and $c_1(\gamma^*)$ is the image of $\gamma^*$ under the composition

$$\tilde{H}^1(\mathbb{C}P^1; \mathcal{C}^\infty(\mathbb{C}^*)) \to \tilde{H}^2(\mathbb{C}P^1; \mathbb{Z}) \to \tilde{H}^2(\mathbb{C}P^1; \mathbb{Z}) \to H^2_{\text{deR}}(\mathbb{C}P^1; \mathbb{C}),$$

$\mathcal{C}^\infty(\mathbb{C}^*) \to \mathbb{C}P^1$ is the sheaf of germs of $\mathbb{C}^*$-valued smooth functions, the first homomorphism is induced by the exponential short exact sequence of sheaves, and the last homomorphism is the de Rham isomorphism (using $\mathbb{C}$ instead of $\mathbb{R}$-coefficients simplifies the computation).
We find a representative $\omega \in E^2(\mathbb{C}P^1)$ for $c_1(\gamma^*), H^2_{\text{deR}}(\mathbb{C}P^1)$ by unwinding the definitions. Let

$U_0 = \{[X_0, X_1] \in \mathbb{C}P^1 : X_0 \neq 0\}, \quad U_1 = \{[X_0, X_1] \in \mathbb{C}P^1 : X_1 \neq 0\},$

be the usual open subsets isomorphic to $\mathbb{C}$. The bundle maps

$\gamma|_{U_0} \xrightarrow{h_0} U_0 \times \mathbb{C}, \quad (\ell, c_0, c_1) \mapsto c_0,$

$\gamma|_{U_1} \xrightarrow{h_1} U_1 \times \mathbb{C}, \quad (\ell, c_0, c_1) \mapsto c_1,$

are the trivializations of $\gamma$ with the overlap map

$h_0 \circ h_1^{-1} : U_0 \cap U_1 \times \mathbb{C} \to U_0 \cap U_1 \times \mathbb{C}, \quad ([X_0, X_1], c_1) \mapsto ([X_0, X_1], c_0 = (X_0/X_1)c_1).$

Thus, the corresponding transition data for $\gamma$ is given by

$U_0 \cap U_1 \to \mathbb{C}^*, \quad [X_0, X_1] \to \frac{X_0}{X_1}.$

The induced transition data for $\gamma^*$ is described by

$g \in \check{Z}^1(\mathcal{O}_U; \mathcal{C}^\infty(\mathbb{C}^*)), \quad g_01([X_0, X_1]) = \frac{X_1}{X_0},$

with $g_{10} = 1/g_01, g_{00}, g_{11} \equiv 1$ (as functions on $U_0 \cap U_0$ and $U_1 \cap U_1$). It determines elements

$[g] \in \check{H}^1(\mathcal{O}_U; \mathcal{C}^\infty(\mathbb{C}^*)), \quad [[g]] \in \check{H}^1(\mathbb{C}P^1; \mathcal{C}^\infty(\mathbb{C}^*))).

The short exact sequence of sheaves inducing the first arrow in the statement of the problem is

$0 \to \mathbb{Z} \to \mathcal{C}^\infty(\mathbb{C}) \xrightarrow{\exp} \mathcal{C}^\infty(\mathbb{C}^*) \to 0$

$f \mapsto e^{2\pi i f}$

In order to find the image of $\gamma^*$ (or equivalently of $[[g]]$) in $\check{H}^2(\mathbb{C}P^1; \mathbb{Z})$, apply the Snake Lemma construction to the diagram

$0 \to \check{C}^2(\mathcal{U}'; \mathbb{Z}) \xrightarrow{i_2} \check{C}^2(\mathcal{U}'; \mathcal{C}^\infty(\mathbb{C})) \xrightarrow{\exp_2} \check{C}^2(\mathcal{U}'; \mathcal{C}^\infty(\mathbb{C}^*)) \to 0$

for a refinement $\mathcal{U}'$ of $\{U_0, U_1\}$. Since $g_{01} \in C^\infty(U_0 \cap U_1; \mathbb{C}^*)$ does not have a well-defined logarithm ($g_{01}$ corresponds to $z \mapsto z$ on $\mathbb{C}^*$ under the usual identification of $U_0$ with $\mathbb{C}$),

$g \in \check{Z}^1(\mathcal{O}_U; \mathcal{C}^\infty(\mathbb{C}^*))) \subset \check{C}^1(\mathcal{O}_U; \mathcal{C}^\infty(\mathbb{C}^*)$)

is not in the image of the homomorphism $\exp_1$. Thus, we need to take a proper refinement $\mathcal{U}'$ of $\{U_0, U_1\}$ and choose a refining map $\mu$. Let

$U'_0 = \{[X_0, X_1] \in \mathbb{C}P^1 : |X_0| > |X_1|\},
U'_+ = U_1 - \{[r, 1] \in U_1 : r \in [1, \infty)\},
U'_- = U_1 - \{[r, 1] \in U_1 : r \in (-\infty, -1)\},
\mathcal{U}' = \{U'_0, U'_+, U'_-\},
\mu : (0, +, -) \to (0, 1, 1).$
Thus, \((\mu^* g)_{0\pm} = g_0|_{U_0 \cap U_{\pm}}, (\mu^* g)_{+\pm} = \pm 1\), and \(\mu^* g = \exp_1(\bar{g})\), with \(\bar{g} \in \check{H}^1(U'; \mathcal{C}^\infty(\mathbb{C}))\) described by
\[
\bar{g}_0 \pm ([X_0, X_1]) = \frac{1}{2\pi i} \ln \left(\frac{X_1}{X_0}\right), \quad \text{Im} \bar{g}_0 = (0, 1), \quad \text{Im} \bar{g}_0 = (-1/2, 1/2),
\]
\[
\bar{g}_\pm = -\bar{g}_0, \quad \bar{g}_0; \bar{g}_\pm \equiv 0.
\]

By the proof of the Snake Lemma, there exists \(h \in \check{Z}^2(U'; \mathbb{Z})\) such that \(i_2(h) = \delta_1(\bar{g})\). By the Snake Lemma construction, the image of \([g]\) in \(\check{H}^1(\mathbb{C}P^1; \mathcal{C}^\infty(\mathbb{C}^*))\) under the boundary homomorphism in the corresponding long exact sequence of modules is \([h]\) in \(\check{H}^2(\mathbb{C}P^1; \mathbb{Z})\).

Via the inclusion \(\mathbb{Z} \longrightarrow \mathbb{C}, \langle [h] \rangle \in \check{H}^2(\mathbb{C}P^1; \mathbb{C})\). It remains to compute its image in \(\check{H}^2_{\text{deR}}(\mathbb{C}P^1; \mathbb{C})\) under the de Rham isomorphism. In this case, this involves going through two boundary homomorphisms. The first arises from the Snake Lemma construction for the diagram
\[
0 \longrightarrow \check{C}^2(U'; \mathbb{C}) \longrightarrow \check{C}^2(U'; \mathcal{C}^\infty(\mathbb{C})) \xrightarrow{d} \check{C}^2(U'; \mathcal{Z}^1) \longrightarrow 0
\]
\[
\delta \downarrow \quad \delta \downarrow \quad \delta \downarrow
\]
\[
0 \longrightarrow \check{C}^1(U'; \mathcal{C}) \longrightarrow \check{C}^1(U'; \mathcal{C}^\infty(\mathbb{C})) \xrightarrow{d} \check{C}^1(U'; \mathcal{Z}^1) \longrightarrow 0
\]
where \(\mathcal{Z}_1 \subset \mathcal{E}_1\) is the sheaf of germs of closed \(\mathbb{C}\)-valued 1-forms. By the previous paragraph, the construction of the Snake Lemma maps the element
\[
\alpha \in \check{Z}^1(U'; \mathcal{Z}^1) \subset \check{C}^1(U'; \mathcal{Z}^1), \quad \alpha_{\star \star} = d\bar{g}_{\star \star},
\]
to \(h\). Let \(\beta \in \check{Z}^1(\{U_0, U_1\}; \mathcal{Z}^1)\) be given by
\[
\beta_{01} \in E^1(U_0 \cap U_1), \quad \beta_{01}(z) = \frac{1}{2\pi i} \frac{dz}{z}, \quad \text{where} \quad z = \frac{X_1}{X_0}.
\]
Since \(\mu^* \beta = \alpha\), the boundary homomorphism for the short exact sequence
\[
0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{C}^\infty(\mathbb{C}) \xrightarrow{d} \mathcal{Z}^1 \longrightarrow 0
\]
takes \([\beta]\) in \(\check{H}^1(\mathbb{C}P^1; \mathcal{Z}^1)\) to \([h]\) in \(\check{H}^2(\mathbb{C}P^1; \mathbb{C})\).

Finally, we need to find a preimage \(\omega \in \check{H}^0(\mathbb{C}P^1; \mathcal{Z}^2) = \mathcal{Z}^2(\mathbb{C}P^1) = E^2(\mathbb{C}P^1)\) of \([\beta]\) under the boundary homomorphism for the short exact sequence
\[
0 \longrightarrow \mathcal{Z}^1 \longrightarrow \mathcal{E}^1 \xrightarrow{d} \mathcal{Z}^2 \longrightarrow 0
\]
of sheaves over \(\mathbb{C}P^1\). This involves applying the Snake Lemma to the diagram
\[
0 \longrightarrow \check{C}^1(\{U_0, U_1\}; \mathcal{Z}^1) \longrightarrow \check{C}^1(\{U_0, U_1\}; \mathcal{E}^1) \xrightarrow{d} \check{C}^1(\{U_0, U_1\}; \mathcal{Z}^2) \longrightarrow 0
\]
\[
\delta \downarrow \quad \delta \downarrow \quad \delta \downarrow
\]
\[
0 \longrightarrow \check{C}^0(\{U_0, U_1\}; \mathcal{Z}^1) \longrightarrow \check{C}^0(\{U_0, U_1\}; \mathcal{E}^1) \xrightarrow{d} \check{C}^0(\{U_0, U_1\}; \mathcal{Z}^2) \longrightarrow 0
\]
Let $\phi \in C^\infty(\mathbb{C}P^1)$ be such that $\phi([X_0, X_1]) = 1$ if $|X_0| < |X_1|$ and $\phi([X_0, X_1]) = 0$ if $|X_0| > 2|X_1|$. Thus,

$$
\eta \in \check{\mathcal{O}}^0(\{U_0, U_1\}; \mathcal{E}^1), \quad \eta_0 = -\phi \beta_{01} \in E^1(U_0), \quad \eta_1 = (1-\phi) \beta_{01} \in E^1(U_1),
$$

is well-defined. Since $d\eta_0 = d\eta_1$ on $U_0 \cap U_1$, there is a unique 2-form $\omega \in E^2(\mathbb{C}P^1)$ such $\omega|_{U_i} = d\eta_i$ on $U_i$. Since

$$(\delta \eta)_{01} \equiv \eta_1|_{U_0 \cap U_1} - \eta_0|_{U_0 \cap U_1} = \beta_{01},$$

$\omega \in \check{Z}^0(\{U_0, U_1\}; \mathcal{Z}^2)$ is mapped to $\beta$ by the Snake Lemma. Thus,

$$
[w] \in H^2_{\text{deR}}(\mathbb{C}P^1) \equiv \frac{E^2(\mathbb{C}P^1)}{dE^1(\mathbb{C}P^1)} = \frac{\check{H}^0(\mathbb{C}P^1; \mathcal{E}^1)}{d \check{H}^0(\mathbb{C}P^1; \mathcal{E}^1)}
$$

corresponds to $[[\beta]] \in \check{H}^1(\mathbb{C}P^1; \mathcal{Z}^1)$ and $[[h]] \in \check{H}^2(\mathbb{C}P^1; \mathbb{C})$ under the isomorphisms factoring the de Rham isomorphism and to the image of $\gamma^*$. 

Using Stokes' Theorem, we now obtain

$$
\int_{\mathbb{C}P^1} c_1(\gamma^*) = \int_{\mathbb{C}P^1} \omega = \int_{U_0'} \omega = -\frac{1}{2\pi i} \int_{U_0'} d \left( \frac{\phi}{z} \right) = -\frac{1}{2\pi i} \int_{S^1} \phi \frac{dz}{z} = -\frac{1}{2\pi i} \int_{S^1} \frac{dz}{z} = -1.
$$

Remark: With the “correct” definition of $c_1$, the answer should be 1. Thus, $c_1(L)$ should really be defined to be the negative of the image of $L$ under the above composition of homomorphism. In the note for PS9 #2, I repeated a mistake from G&H. Their proof that their incorrect definition of $c_1(L)$ is the correct one (i.e. satisfies 2. in Proposition on p141) contains an error. The relation between $\theta_\alpha$ and $\theta_\beta$ worked out in Section 5 Chapter 0 (the last displayed expression on p72) is the opposite of the third equation in the proof on p141; this would change the sign in the relation. The seemingly natural isomorphism between the Čech and de Rham cohomologies in G&H and Warner is actually not the natural one from a certain perspective. In particular, there is a separate isomorphism on each level, i.e. between $\check{H}^p$ and $H^p_{\text{deR}}$. They can be unified by forming a double complex, $\check{C}^p(\mathbb{U}; \mathcal{E}^q)$, with the differential $\check{D}_{p,q} = \delta + (-1)^p d$, where $\delta$ and $d$ are the usual Čech and de Rham differentials; the sign is needed to insure that $D^2 = 0$. The Čech and de Rham complexes then inject into this double complex, inducing isomorphisms in cohomology. The induced isomorphism between $\check{H}^p$ and $H^p_{\text{deR}}$ is then $(-1)^{p(p+1)/2}$ times the isomorphism in G&H, correcting the sign error in the definition of $c_1(L)$ in the de Rham cohomology.