MAT 531: Topology&Geometry, II
Spring 2011

Final Exam

Instructions

- Give concise proofs, quoting established facts as appropriate; no treatises.
- The problems are worth 20 points each, but are not necessarily of the same difficulty. Parts of a problem may not carry equal weight.
- Your final exam score will be based on your work on 5 problems: 2 (highest-scoring) from Part I, 2 from Part II, and 1 from Part III. The points earned on the bonus problem will be added to your final-exam score, even if the total exceeds 100. However, *NO extra credit will be awarded for solutions to any of the other three problems.*
- Please start each problem on a new sheet of paper. When you are finished, please assemble your solutions in order and attach them to the cover sheet. Do *not* attach this problem sheet.

Part I (*choose 2 problems from 1,2, and 3*)

1. *Let* $f\colon \mathbb{R}P^3 \longrightarrow T^3 \equiv (S^1)^3$ *be a smooth map. Show that* f *is not an immersion.*

2. Let X and Y be the vector fields on $\mathbb{R}^3$ given by

$$X = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\,, \qquad Y = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\,.$$

(a) Compute the flows φ_s and ψ_t of X and Y (give formulas).

(b) Do these flows commute?

3. Let M and N be smooth oriented connected manifolds and $H\colon M\times[0,1] \longrightarrow N$ a smooth map. For each $t\in[0,1]$, define

$$H_t\colon M \longrightarrow N, \qquad H_t(p) = H(p,t).$$

(a) Suppose H_t is a diffeomorphism for every $t\in[0,1]$. Show that H_0 is orientation-preserving if and only if H_1 is.

(b) Suppose instead that M is compact and H_0, H_1 are diffeomorphisms. Show that H_0 is orientation-preserving if and only if H_1 is.

(c) Give an example so that H_0 and H_1 are diffeomorphisms, with H_0 orientation-preserving and H_1 orientation-reversing.

Part II (*choose 2 problems from 4,5, and 6*)

4. Let M be a smooth manifold obtained by identifying two copies of a Mobius Band, M_1 and M_2, along their boundary circles. Compute $H^*_{\mathrm{deR}}(M)$.

5. Let M be a smooth manifold admitting an open cover $\{U_i\}_{i=1,\ldots,m}$ such that every intersection $U_{i_1}\cap\ldots\cap U_{i_k}$ is either empty or diffeomorphic to $\mathbb{R}^n$. Show that

(a) if $m=2$, $H^p_{deR}(M)=0$ for all $p\neq 0$;

(b) if $m\geq 2$, $H^p_{deR}(M)=0$ for all $p\geq m-1$.

6. (a) Explain why $\mathbb{R}P^2\times\mathbb{R}P^4$ is not orientable.
(b) Describe the orientable double cover M of $\mathbb{R}P^2\times\mathbb{R}P^4$.
(c) Determine the de Rham cohomology of M.

Part III (*choose 1 problem from 7 and 8*)

7. Let $V, W \longrightarrow S^1$ be smooth real vector bundles. Show that at least one of the vector bundles

$$V, W, V\oplus W \longrightarrow S^1$$

is orientable.

8. Let $\pi\colon V\longrightarrow M$ be a smooth vector bundle. A connection in V is an $\mathbb{R}$-linear map

$$\nabla\colon \Gamma(M;V)\longrightarrow\Gamma(M;T^*M\otimes V) \quad \text{s.t.} \quad \nabla(fs)=\mathrm{d}f\otimes s+f\nabla s \quad \forall\, f\in C^\infty(M),\ s\in\Gamma(M;V).$$

(a) Show that ∇ is a first-order differential operator.

(b) What is the symbol of ∇?

(c) Under what conditions (on M and/or V) is ∇ elliptic?

Bonus Problem

Let $\gamma\longrightarrow\mathbb{C}P^1$ be the tautological (complex) line bundle. Compute

$$\int_{\mathbb{C}P^1} c_1(\gamma^*),$$

where $\mathbb{C}P^1$ has its canonical orientation as a complex manifold and $c_1(\gamma^*)$ is the image of γ^* under the composition

$$\check{H}^1(\mathbb{C}P^1;\mathfrak{C}^\infty(\mathbb{C}^*)) \longrightarrow \check{H}^2(\mathbb{C}P^1;\underline{\mathbb{Z}}) \longrightarrow \check{H}^2(\mathbb{C}P^1;\underline{\mathbb{C}}) \longrightarrow H^2_{\mathrm{deR}}(\mathbb{C}P^1;\mathbb{C}),$$

$\mathfrak{C}^\infty(\mathbb{C}^*)\longrightarrow\mathbb{C}P^1$ is the sheaf of germs of $\mathbb{C}^*$-valued smooth functions, the first homomorphism is induced by the exponential short exact sequence of sheaves, and the last homomorphism is the de Rham isomorphism (using $\mathbb{C}$ instead of $\mathbb{R}$-coefficients simplifies the computation).