MAT 531: Topology&Geometry, II
Spring 2006

Midterm

Give concise proofs, quoting established facts as appropriate; no treatises. This exam continues on the back.

Problem 1 (15 pts)

Suppose M is a smooth manifold and X and Y are smooth vector fields on M. Show directly from definitions that

$$[X,Y] = -[Y,X].$$

(You can assume that $[X,Y]$ is whatever object it is supposed to be, but do state what you are taking it to be.)

Problem 2 (20 pts)

Show that the topological subspace

$$\big\{(x,y)\in\mathbb{R}^2 : x^3+xy+y^3=1\big\}$$

of $\mathbb{R}^2$ is a smooth curve (i.e. admits a natural structure of smooth 1-manifold with respect to which it is a submanifold of $\mathbb{R}^2$).

Problem 3 (5+15 pts)

Let X be a non-vanishing vector field on $\mathbb{R}^3$, written in coordinates as

$$X(x,y,z) = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z} \qquad \text{for some} \quad f,g,h \in C^\infty(\mathbb{R}^3).$$

(a) Find a one-form α on $\mathbb{R}^3$ so that at each point of $\mathbb{R}^3$ the kernel of α is orthogonal to X, with respect to the standard inner-product on $\mathbb{R}^3$.
(b) Find a necessary and sufficient condition on X so that for every point $p\in\mathbb{R}^3$ there exists a surface $S\subset\mathbb{R}^3$ passing through p which is everywhere orthogonal to X (i.e. S is a smooth two-dimensional submanifold of $\mathbb{R}^3$ and $T_mS\subset T_m\mathbb{R}^3$ is orthogonal to $X(m)$ for all $m\in S$).

Problem 4 (20 pts)

Let $S^2\subset\mathbb{R}^3$ be the unit sphere with its standard smooth structure and orientation. Find

$$\int_{S^2}\big(x_1dx_2\wedge dx_3 + x_2dx_1\wedge dx_3 + x_3dx_1\wedge dx_2\big).$$

Problem 5 (25 pts)

Suppose M and N are smooth manifolds. Show that
(a) if M and N are orientable, then $M \times N$ is orientable;
(b) if M is orientable and nonempty and N is not orientable, then $M \times N$ is not orientable;
(c) if M and N are not orientable, then $M \times N$ is not orientable.