1. Let $X$ be a path-connected topological space and let $(\mathcal{S}_*(X), \partial)$ be the singular chain complex of \textit{continuous} simplices into $X$ with \textit{integer} coefficients. Denote by $H_1(X; \mathbb{Z})$ the corresponding first homology group.

(a) Show that there exists a well-defined surjective homomorphism

$$h: \pi_1(X, x_0) \longrightarrow H_1(X; \mathbb{Z}).$$

(b) Show that the kernel of this homomorphism is the commutator subgroup of $\pi_1(X, x_0)$ so that $h$ induces an isomorphism

$$\Phi: \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)] \longrightarrow H_1(X; \mathbb{Z}).$$

This is the first part of the Hurewicz Theorem.

\textit{Hint:} For each $x \in X$, choose a path from $x_0$ to $x$. Use these paths to turn each 1-simplex into a loop based at $x_0$ and construct a homomorphism

$$\mathcal{S}_1(X) \longrightarrow \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)].$$

Show that it vanishes on $\partial \mathcal{S}_2(X)$, well-defined on $\ker \partial$ (may not be necessary), and its composition with $\Phi$ is the identity on $\pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)]$. Sketch something.

2. (a) Prove \textit{Mayer-Vietoris for Cohomology:} If $M$ is a smooth manifold, $U, V \subset M$ open subsets, and $M = U \cup V$, then there exists an exact sequence

$$0 \longrightarrow H^0_{\text{de R}}(M) \xrightarrow{f_0} H^0_{\text{de R}}(U) \oplus H^0_{\text{de R}}(V) \xrightarrow{g_0} H^0_{\text{de R}}(U \cap V) \xrightarrow{\delta_0} H^1_{\text{de R}}(M) \xrightarrow{f_1} H^1_{\text{de R}}(U) \oplus H^1_{\text{de R}}(V) \xrightarrow{g_1} H^1_{\text{de R}}(U \cap V) \xrightarrow{\delta_1} \cdots$$

where

$$f_i(\alpha) = (\alpha|_U, \alpha|_V) \quad \text{and} \quad g_i(\beta, \gamma) = \beta|_{U \cap V} - \gamma|_{U \cap V}.$$ 

(b) Suppose $M$ is a compact connected orientable $n$-dimensional submanifold of $\mathbb{R}^{n+1}$. Show that $\mathbb{R}^{n+1} - M$ has exactly two connected components. How is the compactness of $M$ used?
3. (a) Show that the inclusion map $S^n \rightarrow \mathbb{R}^{n+1} - 0$ induces an isomorphism in cohomology.
(b) Show that for all $n \geq 0$ and $p \in \mathbb{Z}$,
$$H^p_{\text{de R}}(S^n) \cong \begin{cases} \mathbb{R}^2, & \text{if } p = n = 0; \\ \mathbb{R}, & \text{if } p = 0, n, n \neq 0; \\ 0, & \text{otherwise}. \end{cases}$$

*Hint:* Discuss the $p \leq 0$, $p > n$, $n = 0$ cases separately, before starting an induction on $n$. The case $n = 1$ is the subject of 4.14.
(c) Show that $S^n$ is not a product of two positive-dimensional manifolds.

*Note:* Do *not* use the Kunneth formula, unless you are intending to prove it. However, the cup/wedge product can be used and might be useful here.

4. (a) Use Mayer-Vietoris (not Kunneth formula) to compute $H^*_\text{de R}(T^2)$, where $T^2$ is the two-torus, $S^1 \times S^1$. Find a basis for $H^*_{\text{de R}}(T^2)$; justify your answer.
(b) Let $\Sigma_g$ be a compact connected orientable surface of genus $g$ (donut with $g$ holes). Let $B \subset \Sigma_g$ be a small closed ball or a single point. Relate $H^*_{\text{de R}}(\Sigma_g - B)$ to $H^*_{\text{de R}}(\Sigma_g)$ (do not compute $H^p_{\text{de R}}$ for $p = 1, 2$ explicitly).
(c) Show that
$$H^p_{\text{de R}}(\Sigma_g) = \begin{cases} \mathbb{R}, & \text{if } p = 0, 2; \\ \mathbb{R}^{2g}, & \text{if } p = 1; \\ 0, & \text{otherwise}. \end{cases}$$

*Hint:* Discuss the cases $g = 0, 1$ before starting an induction on $g$. Note that $\Sigma_{g_1 + g_2} \cong \Sigma_{g_1} \# \Sigma_{g_2}$.

5. (a) Suppose $q: \tilde{M} \rightarrow M$ is a regular covering projection with a finite group of deck transformations $G$ (so that $M = \tilde{M}/G$). Show that
$$q^*: H^*_\text{de R}(M) \rightarrow H^*_{\text{de R}}(\tilde{M})^G \equiv \{ \alpha \in H^*_\text{de R}(\tilde{M}): g^* \alpha = \alpha \ \forall \ g \in G \}$$
is an isomorphism. Does the statement continue to hold if $G$ is not assumed to be finite?
(b) Determine $H^*_\text{de R}(K)$, where $K$ is the Klein bottle. Find a basis for $H^*_\text{de R}(K)$; justify your answer.

*Hint:* see Exercise 3 on p454 of Munkres.

6. Chapter 5, #4 (p216)

7. Let $K = \mathbb{Z}$ and let $\pi: \mathcal{S}_0 \rightarrow \mathbb{R}$ be the corresponding skyscraper sheaf, with the only non-trivial stack over $0 \in \mathbb{R}$; see 5.11. What is $\mathcal{S}_0$ as a topological space?

*Hint:* it is something familiar.

**Exercises** (*figure these out, but do not hand them in*): Chapter 5, #11, 13, 16, 17 (pp 216,217); verify Lemma 5.14 (p172). The kernel of the first map in (2) of Lemma 5.14 is denoted by $A'' \ast B$ or $\text{Tor}(A'', B)$ and known as the torsion product of $A''$ and $B$; $A'' \ast B = B \ast A''$. 

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