MAT 531: Topology&Geometry, II
Spring 2010

Problem Set 1
Due on Thursday, 2/4, in class

Give concise, but complete, solutions. The entire problem set should not require more than a few pages.

1. Chapter 1, #2 (p50)

2. Suppose a group G acts properly discontinuously on a smooth n-manifold $\tilde{M}$ by diffeomorphisms. Show that the quotient topological space $M = \tilde{M}/G$ admits a unique smooth structure such that the projection map $\tilde{M} \longrightarrow M$ is a local diffeomorphism.
   Note: this implies that the circle, the infinite Mobius band, the Lens spaces (that are important in 3-manifold topology), the real projective space, and the tautological line bundle over it,
   $$S^1 = \mathbb{R}/\mathbb{Z}, \quad s \sim s+1, \qquad MB = (\mathbb{R}\times\mathbb{R})/\mathbb{Z}, \quad (s,t) \sim (s+1,-t),$$
   $$L(n,k) = S^3/\mathbb{Z}_n, \quad (z_1,z_2) \sim \left(\mathrm{e}^{2\pi \mathrm{i}/n}z_1, \mathrm{e}^{2\pi \mathrm{i}k/n}z_2\right) \in \mathbb{C}^2,$$
   $$\mathbb{R}P^n = S^n/\mathbb{Z}_2, \quad x \sim -x, \qquad \gamma_n = (S^n \times \mathbb{R})/\mathbb{Z}_2, \quad (x,t) \sim (-x,-t),$$
   are smooth manifolds in a natural way (k and n are relatively prime in the definition of $L(n,k)$).

3. (a) Show that the quotient topologies on $\mathbb{C}P^n$ given by $(\mathbb{C}^{n+1}-0)/\mathbb{C}^*$ and S^{2n+1}/S^1 are the same (i.e. the map $S^{2n+1}/S^1 \longrightarrow (\mathbb{C}^{n+1}-0)/\mathbb{C}^*$ induced by inclusions is a homeomorphism).
   (b) Show that $\mathbb{C}P^n$ is a compact topological $2n$-manifold. Furthermore, it admits a structure of a *complex* (in fact, *algebraic*) n-manifold, i.e. it can be covered by charts whose overlap maps, $\varphi_\alpha \circ \varphi_\beta^{-1}$, are holomorphic maps between open subsets of $\mathbb{C}^n$ (and rational functions on $\mathbb{C}^n$).
   Note: you can do this with $n+1$ charts.
   (c) Show that $\mathbb{C}P^n$ contains $\mathbb{C}^n$, with its complex structure, as a dense open subset.

4. Chapter 1, #6 (p50) via 2nd suggested approach

5. Verify that the differential $d\psi$ of a smooth map $\psi\colon M \longrightarrow N$, as defined in 1.22 (p16), is indeed well-defined. In other words, $d\psi(v)$ is a derivation on $\tilde{F}_{\psi(m)}$ for all $v \in T_mM$ and $m \in M$.