MAT 531: Topology&Geometry, II
Spring 2010

Overview

(1) Smooth manifolds, tangent vectors, differentials, immersions, etc. (intrinsically and in local coordinates): PS1 #1-5; PS2 #1,2,7; MT06 #1; MT #1,2; FE06 #1

(2) Vector Bundles: PS2 #3-6; PS4 #5; PS5 #4; PS6 #4,5ab; PS8 #2,3; MT06 #5; MT #5; FE06 #3,7

- Set of isomorphism classes of (smooth) real vector bundles of rank k on a paracompact topological space (smooth manifold) is $\check{H}^1(M;\mathrm{GL}_k\mathbb{R})$. If $k=1$ (line bundles), this is an abelian group isomorphic to $H^1(M;\mathbb{Z}_2)$. The sum in $H^1(M;\mathbb{Z}_2)$ corresponds to tensor product of line bundles (multiplication of transition data); the inverse of a line bundle is its dual. In particular, the square of every **real** line is trivial.
- Set of isomorphism classes of (smooth) complex vector bundles of rank k on a paracompact topological space (smooth manifold) is $\check{H}^1(M;\mathrm{GL}_k\mathbb{C})$. If $k=1$ (line bundles), this is an abelian group isomorphic to $H^2(M;\mathbb{Z})$. The sum in $H^2(M;\mathbb{Z})$ corresponds to tensor product of line bundles (multiplication of transition data); the inverse of a line bundle is its dual. The square of a complex line is usually **not** trivial (as a complex line bundle).

(3) Differentials, Inverse FT, Slice Lemma, Implicit FT (I&II): PS3 #1-3; MT06 #2; MT #2

(4) Flows of vector fields, Lie Bracket, Lie Derivative: PS3 #4,5; PS4 #1,2; PS6 #8a; MT06 #1; FE06 #2

- compute flow of a vector field and Lie derivative from the flow: PS3 #6; PS6 #1
- compute the Lie bracket of two vector fields: PS4 #3.

(5) The Differential $d\colon E^p(M)\longrightarrow E^{p+1}(M)$, Frobenius Theorem (I&II), Strong Slice Statement: PS4 #3; PS5 #1-3,5,6; MT06 #3; MT #3

- when does a collection of k vector fields form a subset of coordinate vectors or at least has the same span at each point as the first k coordinate vectors?
- when can a 1-form be written with fewer pieces after a change of coordinates?

(6) de Rham cochain complex, Poincare Lemma, Stokes' Theorem (I&II), and group actions: PS6 #2,3,6b,9; PS7 #5; PS10 #3; MT06 #4; MT #4; FE06 #1,6

- If $\pi:\tilde{M}\longrightarrow M$ is a **regular** covering projection and G is its group of deck transformations, then the homomorphism

$$\pi^*\colon E^*(M)\longrightarrow E^*(\tilde{M})^G \equiv \big\{\tilde{\alpha}\in E^*(\tilde{M})\colon\ g^*\tilde{\alpha}=\tilde{\alpha}\ \forall\, g\in G\big\}$$

is an isomorphism. If in addition, G is **finite**, then the homomorphism

$$\pi^*: H^*_{\text{deR}}(M) \longrightarrow H^*_{\text{deR}}(\tilde{M})^G \equiv \{\tilde{\alpha} \in H^*_{\text{deR}}(\tilde{M}) : g^*[\tilde{\alpha}] = [\tilde{\alpha}] \, \forall \, g \in G\}$$

is also an isomorphism. The cohomology homomorphism fails to be an isomorphism for the simplest non-trivial covering map with G being infinite: $\mathbb{R} \longrightarrow S^1$.

- A covering projection $\pi : \tilde{M} \longrightarrow M$ is regular if the group of deck transformations (maps $g: \tilde{M} \longrightarrow \tilde{M}$ such that $\pi = \pi \circ g$) acts transitively on the fibers of π. Every double (2-to-1) cover is necessarily regular: $G = \mathbb{Z}_2$ with $(-1) \in \mathbb{Z}_2$ interchanging the two points in each fiber of π. For any covering map, the homomorphism

$$\pi_*: \pi_1(\tilde{M}, \tilde{x}_0) \longrightarrow \pi_1(M, x_0), \qquad x_0 = \pi(\tilde{x}_0),$$

is injective. If M and $\tilde{M}$ are connected, π is a regular covering if and only if the image of π_* is a normal subgroup of $\pi_1(M, x_0)$, i.e. preserved by conjugation in $\pi_1(M, x_0)$. So if $\pi_1(M, x_0)$ is abelian, then every covering is regular. Every double cover being regular corresponds to every subgroup $H \subset G$ of index 2, i.e. $|G/H| = 2$, being normal. If G is a group acting on $\tilde{M}$ properly discontinuously (by diffeomorphisms), then the quotient map

$$\pi: \tilde{M} \longrightarrow M = \tilde{M}/G$$

is a regular covering (and π is a smooth map). If G is **finite** and acts on $\tilde{M}$ without fixed points ($g\tilde{x} = \tilde{x}$ for some $\tilde{x} \in \tilde{M}$ if and only if $g = id$), then G acts properly discontinuously and thus $\pi : \tilde{M} \longrightarrow \tilde{M}/G$ is a regular covering and the cohomology of $\tilde{M}/G$ can be computed from the cohomology of $\tilde{M}$ (but not the other way around).

(7) Orientability of manifolds and vector bundles, relations with topology and covering maps: PS6 #4-7,8bc; MT06 #5; FE06 #5,7

(8) Singular chain complex, Hurewicz Theorem: PS7 #1; PS9 #2; FE06 #4

(9) (Co)chain complexes and (co)homology, duals, coefficient changes, Snake Lemma

- Mayer-Vietoris for de Rham cohomology, singular homology, compactly supported cohomology: PS7 #2-4, PS11 #2; FE06 #6b,BP
- Sheafs and Čech Cohomology: PS7 #6,7; PS8 #1-3; PS9 #1,3
- Cohomology from fine resolutions: de Rham Theorem
- Compactly supported cohomology: PS11 #3

(10) Geometric Analysis and Hodge Theory

- Differential operators, symbol, elliptic operators: FE06 #8

- Sobolev Lemma, Rellich Lemma, Fundamental Inequality: PS 10, #4,5
- Laplacian: PS4 #4; PS10 #1,2
- Hodge Decomposition Theorem, Poincare Duality, finite-dimensionality of de Rham cohomology, Kunneth Formula: PS10 #5
- The main point of Chapter 6 is that $H^p_{\text{deR}}(M) \approx \mathcal{H}^p(M)$ for a **compact** (Riemannian) manifold M. While $H^p_{\text{deR}}(M)$ is a quotient of a subspace of $E^p(M)$ (the subspace $\ker d_p$), $\mathcal{H}^p(M)$ is an actual subspace of $\ker d_p$ and the isomorphism to $H^p_{\text{deR}}(M)$ is given by the quotient projection map. One drawback of $\mathcal{H}^p(M)$ is that it depends on the choice of Riemannian metric, but this is not a problem for many applications (such as Poincare Duality and Kunneth formula); more applications will be done in MAT 545 (see also Figure 1 below). If M is not compact, it is generally not true $H^p_{\text{deR}}(M)$ is isomorphic $\mathcal{H}^p(M)$; for example, the space of harmonic functions on $\mathbb{R}^2$ is infinite-dimensional (the real and imaginary parts of a holomorphic function on $\mathbb{C}$ are harmonic), even though $H^0_{\text{deR}}(\mathbb{R}^2) \approx \mathbb{R}$ consists of just the constant functions.

(11) Computing de Rham cohomology of n-manifold M:

- $H^0_{\text{deR}}(M)$; $H^n_{\text{deR}}(M)$ (M orientable/not, compact/not): PS10 #3; PS11 #1
- $H^1_{\text{deR}}(M)$ from $\pi_1(M)$; then $H^{n-1}(M)$ if M is compact orientable
- if $M = \tilde{M}/G$, where G is **finite** and acts freely on $\tilde{M}$, can compute $H^*_{\text{deR}}(M)$ from $H^*_{\text{deR}}(\tilde{M})$ (see above); if G is infinite and acts properly discontinuously on $\tilde{M}$, may be able to compute $\pi_1(M)$ from $\pi_1(\tilde{M})$ (e.g. $\pi_1(M) = G$ if $\pi_1(\tilde{M}) = \{1\}$), but **not** $H^*_{\text{deR}}(M)$ from $H^*_{\text{deR}}(\tilde{M})$
- Mayer-Vietoris (need **open** sets; path-connected not necessarily, unlike van Kampen)
- $H^*_{\text{deR}}(\mathbb{R}^n)$, $H^*_{\text{deR}}(S^n)$, $H^*_{\text{deR}}(\Sigma_g)$: PS7 #3,4

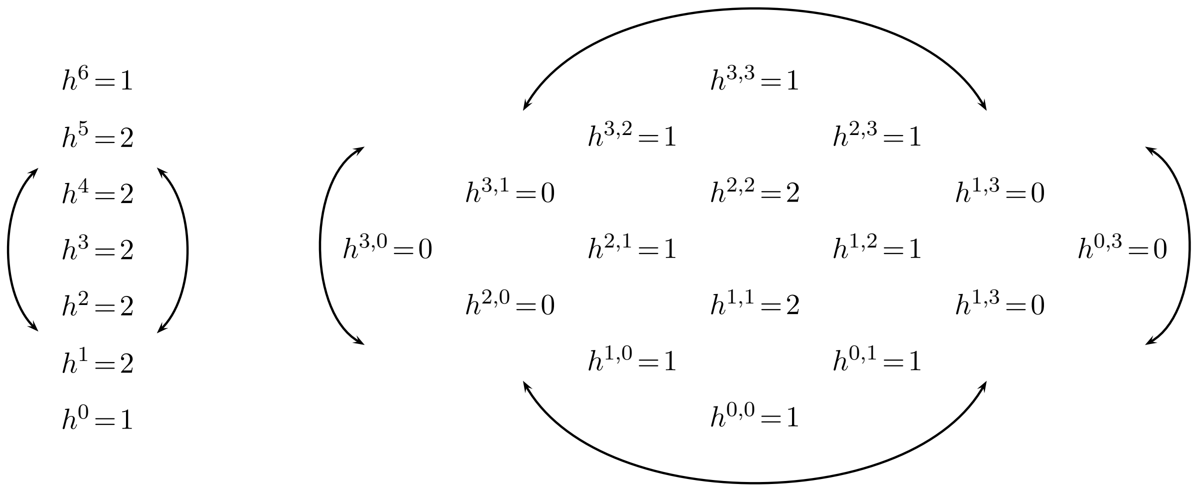

Figure 1: For a compact oriented n-manifold M, Hodge theory leads to Poincare duality; it implies that the dimensions of the de Rham cohomology groups of M are symmetric about $n/2$. For a compact Kahler manifold M (subject of MAT 545), Hodge theory leads to a two-way symmetry, known as Hodge diamond; it implies that the dimensions of the odd cohomologies of a Kahler manifold are even (a quick way to see which manifolds do not admit a Kahler structure). The two diagrams above show the "de Rham segment" and the Hodge diamond for $\mathbb{C}P^2 \times T^2$, which is a real 6-manifold and a Kahler 3-manifold, and their symmetries.

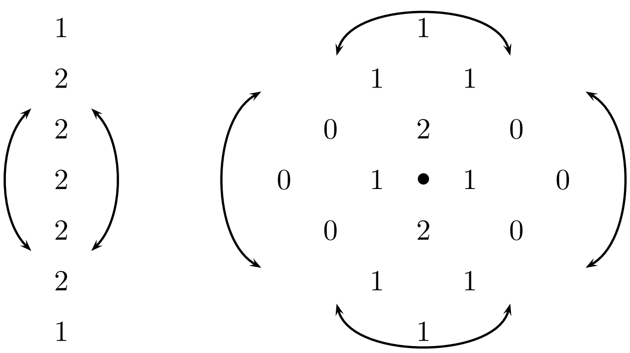

Figure 2: Same diagram as above, but just with numbers; the diamond is symmetric about the center •