1. Suppose $M$ is a compact manifold and $\alpha$ is a nowhere-zero closed one-form. Show that $[\alpha] \neq 0 \in H^1_{\text{der}}(M)$.

We need to show that $\alpha$ is not exact, i.e. $\alpha \neq df$ for any $f \in C^\infty(M)$. Suppose $f : M \to \mathbb{R}$ is a smooth function. Since $M$ is compact, $f$ must achieve its maximum at some point $m \in M$ (not necessarily unique). Then, $df|_m = 0$, since in local coordinates near $m$ all partial derivatives of $f$ must vanish at $m$. Since $\alpha|_m \neq 0$, $df \neq \alpha$.

2. Let $Y$ and $Z$ be the vector fields on $\mathbb{R}^3$ given by

$$Y(x_1, x_2, x_3) = \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} \quad \text{and} \quad Z(x_1, x_2, x_3) = a \frac{\partial}{\partial x_1} + b \frac{\partial}{\partial x_2} + c \frac{\partial}{\partial x_3},$$

where $a$, $b$, and $c$ are constants.

(a) Compute the Lie bracket $[Y, Z]$.
(b) Describe the flows $\varphi_s$ of $Y$ and $\psi_t$ of $Z$.
(c) For what constants $a$, $b$, and $c$ do these two flows commute?

(a) The Lie bracket of two smooth vector fields $Y$ and $Z$ on a manifold $M$ is defined by

$$[Y, Z] : C^\infty(M) \to C^\infty(M), \quad [Y, Z]f = Y(Zf) - Z(Yf).$$

Since the Lie bracket of two coordinate vector fields vanishes, in this case we obtain

$$[Y, Z] = \left( Y(a) \frac{\partial}{\partial x_1} + Y(b) \frac{\partial}{\partial x_2} + Y(c) \frac{\partial}{\partial x_3} \right) - \left( Z(1) \frac{\partial}{\partial x_1} + Z(x_3) \frac{\partial}{\partial x_2} - Z(x_2) \frac{\partial}{\partial x_3} \right)$$

$$= 0 - \left( 0 + c \frac{\partial}{\partial x_2} - b \frac{\partial}{\partial x_3} \right) = -c \frac{\partial}{\partial x_2} + b \frac{\partial}{\partial x_3}.$$

(b) Since $Z$ is a constant vector field, its flow $\psi_t$ are translations with the velocity $(a, b, c)$, i.e.

$$\psi_t(x_1, x_2, x_3) = (x_1 + at, x_2 + bt, x_3 + ct).$$

If $(a, b, c) = 0$, $\psi_t$ is just the identity map, i.e. the flow is stationary. The effect of the flow $\varphi_s$ of $Y$ on the $x_2$ and $x_3$-coordinates is the clockwise rotation by angle $s$. To see that this rotation is clockwise, notice that

$$Y(0, 1, 0) = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3},$$

i.e. points down in $x_3$ at $(x_2, x_3) = (1, 0)$. The first component of $Y$ shifts the planes $x_1 = \text{const}$ at the unit speed in the direction of increasing $x_1$. Thus, $\varphi_s(x_1, x_2, x_3)$ is the clockwise spiral around the $x_1$-axis moving to the right. Explicitly,

$$\varphi_s(x_1, r \cos \theta, r \sin \theta) = \varphi_s(x_1 + s, r \cos(\theta - s), r \sin(\theta - s)).$$
(c) By Problem 2 on PS4, the flows \( \varphi_s \) and \( \psi_t \) for \( Y \) and \( Z \) commute if and only if \( [Y, Z] = 0 \). By part (a), this is the case if and only if \( b = c = 0 \), so that \( \psi_t \) is a sideways translation (i.e. in \( x_1 \) only).

Alternatively, \( \varphi_s \) is a composition of a nontrivial (if \( s \neq 0 \)) sideways translation (i.e. in \( x_1 \)) and a nontrivial rotation in \( x_2 \) and \( x_3 \). On the other hand, \( \psi_t \) is a composition of a sideways translation and a translation in \( x_2 \) and \( x_3 \), either of which might be trivial. Since all translations commute and \((x_2, x_3)\)-rotations commute with \( x_1 \)-translations, the flows \( \varphi_s \) and \( \psi_t \) commute if and only if the nontrivial \((x_2, x_3)\)-rotation for \( \varphi_s \) commutes with the \((x_2, x_3)\)-translation for \( \psi_t \). Since rotations and translations in \( \mathbb{R}^2 \) do not commute, the latter is the case if and only if the \((x_2, x_3)\)-translation for \( \psi_t \) is trivial, i.e. \( b = c = 0 \).

3. Describe explicitly trivializations and transition data for the vector bundle \( TS^2 \to S^2 \).

Suppose \( M \) is a smooth manifold and \( \pi : V \to M \) is a vector bundle of rank \( k \). A trivialization of \( V \) over an open subset \( U \) of \( M \) can be constructed by finding \( s_1, \ldots, s_k \in \Gamma(U; V) \) such that \( s_1(x), \ldots, s_k(x) \in V_x \) are linearly independent vectors (and thus a basis for \( V_x \)) for all \( x \in U \). Such sections \( s_1, \ldots, s_k \) must exist if \( U \) is sufficiently small. Then, for every \( x \in U \) and \( v \in V_x \), there exist unique

\[
c_1(v), \ldots, c_k(v) \in \mathbb{R} \quad \text{s.t.} \quad v = c_1(v)s_1(x) + \ldots + c_k(v)s_k(x).
\]

A trivialization of \( V \) over \( U \) can then be defined by

\[
h : V|_U \to U \times \mathbb{R}^k, \quad h(v) = (\pi(x), c_1(v), \ldots, c_k(v)).
\]

If \( h_\alpha : V|_{U\alpha} \to U\alpha \times \mathbb{R}^k \) and \( h_\beta : V|_{U\beta} \to U\beta \times \mathbb{R}^k \) are two trivializations, then

\[
h_\alpha \circ h_\beta^{-1} : (U\alpha \cap U\beta) \times \mathbb{R}^k \to (U\alpha \cap U\beta) \times \mathbb{R}^k
\]

is a diffeomorphism which commutes with the projection map onto the first component and is linear on the fibers of this projection. Thus, for every \( x \in U\alpha \cap U\beta \) there exists a unique \( g_{\alpha\beta}(x) \in \text{GL}_k \mathbb{R} \) such that

\[
\{h_\alpha \circ h_\beta^{-1}\}(x, v) = (x, g_{\alpha\beta}(x) \cdot v) \quad \forall \ v \in \mathbb{R}^k.
\]

The smooth map

\[
g_{\alpha\beta} : U\alpha \cap U\beta \to \text{GL}_k \mathbb{R}
\]

is then the transition map from \((U\beta, h_\beta)\) to \((U\alpha, h_\alpha)\).

Suppose next that \( V = TM \) is the tangent bundle. If

\[
\varphi_\alpha = (x_1, \ldots, x_n) : U\alpha \to \mathbb{R}^n
\]

is a coordinate chart, then the coordinate tangent vectors

\[
\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \in \Gamma(U\alpha; TM)
\]

are linearly independent at each point of \( U \). Furthermore, for all \( x \in U \) and \( v \in T_x M \),

\[
v = v(x_1) \frac{\partial}{\partial x_1} \bigg|_x + \ldots + v(x_n) \frac{\partial}{\partial x_n} \bigg|_x.
\]
It is sufficient to check this identity on the functions \( x_1, \ldots, x_n \) on \( U \) as they form a basis for \( F_x/F_x^2 \) for all \( x \in U \); see Theorem 1.17 of Warner. Thus, we obtain a trivialization of \( TM \) over \( U_\alpha \):

\[
h_\alpha : TM|_{U_\alpha} \rightarrow U_\alpha \times \mathbb{R}^n, \quad h_\alpha(v) = (\pi(v), v(x_1), \ldots, v(x_n)).
\]

Suppose

\[
\varphi_\beta = (y_1, \ldots, y_n) : U_\alpha \rightarrow \mathbb{R}^n
\]

is another coordinate chart. By the above,

\[
\frac{\partial}{\partial y_j} = \sum_{i=1}^{n} \left( \frac{\partial}{\partial y_j} x_i \right) \frac{\partial}{\partial x_i} = \sum_{i=1}^{n} J(\varphi_\alpha \circ \varphi_\beta^{-1})_{ij} \frac{\partial}{\partial x_i}
\]

and

\[
\sum_{i=1}^{n} v(x_i) \frac{\partial}{\partial x_i} = v = \sum_{j=1}^{n} v(y_j) \frac{\partial}{\partial y_j} = \sum_{j=1}^{n} \left( \sum_{i=1}^{n} J(\varphi_\alpha \circ \varphi_\beta^{-1})_{ij} \right) \frac{\partial}{\partial x_i}
\]

where \( J(\varphi_\alpha \circ \varphi_\beta^{-1}) \) is the Jacobian of the map \( \varphi_\alpha \circ \varphi_\beta^{-1} \) between open subsets of \( \mathbb{R}^n \). Thus,

\[
(v(x_1), \ldots, v(x_n))^t = J(\varphi_\alpha \circ \varphi_\beta^{-1})(v(y_1), \ldots, v(y_n))^t \quad \Rightarrow \quad g_{\alpha \beta}(x) = J(\varphi_\alpha \circ \varphi_\beta^{-1})|_{\varphi_\beta(x)}.
\]

In other words, the transition map from \((U_\beta, h_\beta)\) to \((U_\alpha, h_\alpha)\) is \( g_{\alpha \beta} = J(\varphi_\alpha \circ \varphi_\beta^{-1}) \circ \varphi_\beta \).

In the present case, \( M = S^2 \subset \mathbb{R}^3 \). We take

\[
U_\alpha = S^2 - (0, 0, 1), \quad U_\beta = S^2 - (0, 0, -1),
\]

and \( \varphi_\alpha : U_\alpha \rightarrow \mathbb{R}^2 \) and \( \varphi_\beta : U_\beta \rightarrow \mathbb{R}^2 \) to be the projections from the north and south poles. In other words, for each \((x, y, z) \in U_\alpha\), \( \varphi_\alpha(x, y, z) \in \mathbb{R}^2 \) is the intersection of the line passing through \((0, 0, 1)\) and \((x, y, z)\) with the \(xy\)-plane. Explicitly,

\[
\varphi_\alpha(x, y, z) = \left( \frac{x}{1-z}, \frac{y}{1-z} \right) \quad \text{and} \quad \varphi_\beta(x, y, z) = \left( \frac{x}{1+z}, \frac{y}{1+z} \right).
\]

Thus,

\[
g_{\alpha \beta} = J(\varphi_\alpha \circ \varphi_\beta^{-1}) \circ \varphi_\beta = \frac{1}{(s^2 + t^2)^2} \begin{pmatrix} -s^2 + t^2 & 2st \\ -2st & s^2 - t^2 \end{pmatrix} = \begin{pmatrix} (1+z)^2 & -2xy \\ -2xy & x^2 - y^2 \end{pmatrix}.
\]

Thus, the transition data for the trivializations \( \{(U_\alpha, h_\alpha), (U_\beta, h_\beta)\} \) of \( TS^2 \rightarrow S^2 \) is

\[
g_{\alpha \beta}, g_{\beta \alpha} = g_{\alpha \beta}^{-1} : U_\alpha \cap U_\beta \rightarrow GL_2 \mathbb{R},
\]

\[
g_{\alpha \beta}(x, y, z) = \frac{(1+z)^2}{(x^2 + y^2)^2} \begin{pmatrix} -x^2 + y^2 & -2xy \\ -2xy & x^2 - y^2 \end{pmatrix},
\]

\[
g_{\beta \alpha}(x, y, z) = \frac{1}{(1+z)^2} \begin{pmatrix} -x^2 + y^2 & -2xy \\ -2xy & x^2 - y^2 \end{pmatrix} = \frac{(1-z)^2}{(x^2 + y^2)^2} \begin{pmatrix} -x^2 + y^2 & -2xy \\ -2xy & x^2 - y^2 \end{pmatrix}.
\]
since \(x^2 + y^2 + z^2 = 1\).

Alternatively, we can view \(S^2\) as the complex one-manifold

\[
\mathbb{CP}^1 = \{[X_0, X_1] : (X_0, X_1) \in \mathbb{C}^2 \setminus \{0\}\},
\]

or the Riemann sphere. We then take

\[
U_0 = \{[X_0, X_1] \in \mathbb{CP}^1 : X_0 \neq 0\}, \quad U_1 = \{[X_0, X_1] \in \mathbb{CP}^1 : X_1 \neq 0\},
\]

and identify \(\varphi_0(\{X_0, X_1\}) = X_1/X_0, \quad \varphi_1(\{X_0, X_1\}) = X_0/X_1\)

\[
\Rightarrow \varphi_0 \circ \varphi_1^{-1} : \mathbb{C} \longrightarrow \mathbb{C}^*, \quad \{(\varphi_0 \circ \varphi_1^{-1})(z) = z^{-1}, \ g_{01} = \mathcal{J}(\varphi_0 \circ \varphi_1^{-1}) \circ \varphi_1 = -1/z^2 = -X_1^2/X_0^2\}.
\]

Thus, the transition data for the trivializations \(\{(U_0, h_0), (U_1, h_1)\}\) of the complex line bundle \(T\mathbb{CP}^1 \longrightarrow \mathbb{CP}^1\)
is given by

\[
g_{01}, g_{10} = g_{01}^{-1} : U_0 \cap U_1 \longrightarrow \text{GL}_1 \mathbb{C} = \mathbb{C}^*, \quad g_{01}(\{X_0, X_1\}) = -X_1^2/X_0^2, \quad g_{10}(\{X_0, X_1\}) = -X_0^2/X_1^2.
\]

Forgetting the complex structure in the complex line bundle \(TCP^1 \longrightarrow \mathbb{CP}^1\), we obtain transition data for the real bundle \(TS^2 \longrightarrow S^2\) of rank 2 by viewing \(\mathbb{C}^*\) as subspace of \(\text{GL}_2 \mathbb{R}\). An explicit (and standard in complex analysis) identification of \(S^2\) with \(\mathbb{CP}^1\) is given by

\[
S^2 \longrightarrow \mathbb{CP}^1, \quad (x, y, z) \longrightarrow [x + iy, 1 + z] = [1 - z, x - iy],
\]
i.e. by extending the chart \(\varphi_\beta\). Since \(U_\alpha = U_0\) and \(U_\beta = U_1\) with this identification, plugging this into the above transition data we obtain

\[
g_{\alpha\beta} = g_{01}, \ g_{\beta\alpha} = g_{10} : U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^* \subset \text{GL}_2 \mathbb{R},
\]

\[
g_{\alpha\beta}(x, y, z) = -\left(\frac{1+z}{x+iy}\right)^2 = -\left(\frac{1+z}{x^2+y^2}\right)^2 \left((x^2-y^2)^2 - 2xy\right) = \left(\frac{1+z}{x^2+y^2}\right)^2 \left(-x^2+y^2 \quad 2xy \quad -x^2+y^2\right),
\]

\[
g_{\beta\alpha}(x, y, z) = -\left(\frac{1-z}{x-iy}\right)^2 = -\left(\frac{1-z}{x^2+y^2}\right)^2 \left((x^2-y^2)^2 + 2xy\right) = \left(\frac{1-z}{x^2+y^2}\right)^2 \left(-x^2+y^2 \quad 2xy \quad -x^2+y^2\right).
\]

This transition data differs from \(g_{\alpha\beta}\) and \(g_{\beta\alpha}\) obtained previously, because under the above identification of \(S^2\) with \(\mathbb{CP}^1\) the chart \((U_\alpha, \varphi_\alpha)\) defined previously corresponds to the chart \((U_0, \varphi_0)\), while \((U_\beta, \varphi_\beta)\) corresponds to \((U_1, \varphi_1)\). 

**Part II (choose 2 problems from 4,5, and 6)**

4. **Compute the singular homology of a point directly from the definition.**

If \(R\) is any ring (e.g. \(\mathbb{R}\)), we will show that

\[
H_p(pt; R) \approx \begin{cases} R, & \text{if } p = 0; \\ 0, & \text{otherwise}. \end{cases}
\]
Let \( S_p(pt) \equiv S_p(X; R) \) be the free \( R \)-module with a basis consisting of smooth maps \( f: \Delta^p \to pt \), where \( \Delta^p \subset \mathbb{R}^p \) is the standard \( p \)-simplex. In this case, there is only one such map, which we denote by \( f_p \). Thus,

\[
S_p(pt) = R\{f_p\} \cong R \quad \forall \ p \geq 0.
\]

By definition, \( H_p(pt; R) \) is the \( p \)-th homology of the chain complex \((S_p(pt), \partial)\), where \( \partial_p: S_p(pt) \to S_{p-1}(pt) \) is the boundary operator.

If \( p \geq 1 \),

\[
\partial_p f_p = \sum_{i=0}^{i=p} (-1)^i f_p \circ \iota_i^p,
\]

where \( \iota_i^p: \Delta^p \to \Delta^p \) are the maps described in Section 4.6 of Warner. However, in this case it does not matter what these maps are, since the map \( f_p \circ \iota_i^p: \Delta^{p-1} \to pt \) must be \( f_{p-1} \). Thus,

\[
\partial_p f_p = \sum_{i=0}^{i=p} (-1)^i f_p \circ \iota_i^p = \sum_{i=0}^{i=p} (-1)^i f_{p-1} = \begin{cases} 0, & \text{if } p \text{ is odd;} \\ f_{p-1}, & \text{if } p > 0 \text{ is even.} \end{cases}
\]

By definition, \( \partial_0 \equiv 0 \). Therefore, the homomorphism

\[
\partial_p: S_p(pt) \to S_{p-1}(pt)
\]

is an isomorphism if \( p > 0 \) is even and trivial otherwise. We conclude that

\[
\ker \partial_p \cong \begin{cases} \mathbb{R}, & \text{if } p > 0 \text{ is odd or } p = 0; \\ 0, & \text{otherwise;} \end{cases} \quad \text{Im} \partial_{p+1} \cong \begin{cases} \mathbb{R}, & \text{if } p > 0 \text{ is odd;} \\ 0, & \text{otherwise;} \end{cases}
\]

\[
\implies H_p(pt; R) \equiv \ker \partial_p / \text{Im} \partial_{p+1} \cong \begin{cases} \mathbb{R}, & \text{if } p = 0; \\ 0, & \text{otherwise.} \end{cases}
\]

5. (a) Show that \( \mathbb{R}P^2 \times \mathbb{R}P^3 \times \mathbb{R}P^4 \) is not orientable.

(b) Describe the orientable double cover of \( \mathbb{R}P^2 \times \mathbb{R}P^3 \times \mathbb{R}P^4 \).

(a) By Problem 5 on the 06 midterm,

\[
\mathbb{R}P^2 \times \mathbb{R}P^3 \times \mathbb{R}P^4 = \mathbb{R}P^2 \times (\mathbb{R}P^3 \times \mathbb{R}P^4)
\]

is orientable if and only if \( \mathbb{R}P^2 \) and \( \mathbb{R}P^3 \times \mathbb{R}P^4 \) are orientable. On the other hand, by Problem 6b on PS6, \( \mathbb{R}P^n \) is orientable if and only if \( n \) is odd. Thus, \( \mathbb{R}P^2 \) is not orientable, and neither is \( \mathbb{R}P^2 \times \mathbb{R}P^3 \times \mathbb{R}P^4 \).

Alternatively, let

\[
\pi: S^2 \times S^3 \times S^4 \to \mathbb{R}P^2 \times \mathbb{R}P^3 \times \mathbb{R}P^4
\]
be the natural covering map. It is a regular covering map and the group of deck transformations is the subgroup $G$ of diffeomorphisms of $S^2 \times S^3 \times S^4$ generated by the diffeomorphisms
\[ a_2 \times \text{id}_{S^3} \times \text{id}_{S^4}, \quad \text{id}_{S^2} \times a_3 \times \text{id}_{S^4}, \quad \text{id}_{S^2} \times \text{id}_{S^3} \times a_4, \]
where
\[ a_n : S^n \to S^n, \quad a_n(x) = -x, \]
is the antipodal map. In particular, $G$ acts without fixed points and
\[ \mathbb{R}P^2 \times \mathbb{R}P^3 \times \mathbb{R}P^4 = (S^2 \times S^3 \times S^4) / G. \]
Since $a_n$ is orientation-preserving if and only if $n$ is odd (see Problem 6a on PS6), the diffeomorphism $a_2 \times \text{id}_{S^3} \times \text{id}_{S^4}$ is orientation-reversing. Therefore, $\mathbb{R}P^2 \times \mathbb{R}P^3 \times \mathbb{R}P^4$ is not orientable.

(b) Let $a_n$ and $G$ be as in the previous paragraph. Denote by $G_0 \subset G$ the subgroup consisting of orientable diffeomorphisms. The oriented double cover of the non-orientable space
\[ \mathbb{R}P^2 \times \mathbb{R}P^3 \times \mathbb{R}P^4 = (S^2 \times S^3 \times S^4) / G \]
is given by
\[ \tilde{M} = (S^2 \times S^3 \times S^4) / G_0. \]
Since $\text{id}_{S^2} \times a_3 \times \text{id}_{S^4}$ is orientation-preserving, while $a_2 \times \text{id}_{S^3} \times \text{id}_{S^4}$ and $\text{id}_{S^2} \times \text{id}_{S^3} \times a_4$ are orientation-reversing,
\[ G_0 = \{ \text{id}_{S^2} \times \text{id}_{S^3} \times \text{id}_{S^4}, \text{id}_{S^2} \times a_3 \times \text{id}_{S^4}, a_2 \times \text{id}_{S^3} \times a_4, a_2 \times a_3 \times a_4 \} \]
\[ = \{ \text{id}_{S^2} \times \text{id}_{S^3} \times \text{id}_{S^4}, \text{id}_{S^2} \times a_3 \times \text{id}_{S^4} \} \times \{ \text{id}_{S^2} \times \text{id}_{S^3} \times \text{id}_{S^4}, a_2 \times \text{id}_{S^3} \times a_4 \}. \]
Thus,
\[ \tilde{M} = (S^2 \times S^3 \times S^4) / G_0 = (S^2 \times \mathbb{R}P^3 \times S^4) / G_0', \]
where
\[ G_0' = \{ \text{id}_{S^2} \times \text{id}_{\mathbb{R}P^3} \times \text{id}_{S^4}, a_2 \times \text{id}_{\mathbb{R}P^3} \times a_4 \}. \]

6. (a) **Determine the de Rham cohomology of $\mathbb{R}P^2$.**

(b) **Determine the de Rham cohomology of $\mathbb{R}P^2 \# \mathbb{R}P^2$.**

(a) Since $\mathbb{R}P^2$ is connected,
\[ H^0_{\text{deR}}(\mathbb{R}P^2) \approx \mathbb{R}. \]
Since $\mathbb{R}P^2$ is a connected and non-orientable 2-manifold,
\[ H^2_{\text{deR}}(\mathbb{R}P^2) = 0. \]
Since $\mathbb{R}P^2$ is connected and $\pi_1(\mathbb{R}P^2) \approx \mathbb{Z}_2$, by Hurewicz and de Rham Theorems
\[ H_1(\mathbb{R}P^2; \mathbb{Z}) \approx \text{Abel}(\pi_1(\mathbb{M})) \approx \mathbb{Z}_2 \quad \implies \quad H_1(\mathbb{R}P^2; \mathbb{R}) \approx H_1(\mathbb{R}P^2; \mathbb{Z}) \otimes_\mathbb{Z} \mathbb{R} = 0 \]
\[ \implies \quad H^1_{\text{deR}}(\mathbb{R}P^2) \approx H^1(\mathbb{R}P^2; \mathbb{R}) \approx (H_1(\mathbb{R}P^2; \mathbb{R}))^* = 0. \]
Since $\mathbb{R}P^2$ is a two-dimensional manifold, all other de Rham groups are zero.

Alternatively, let $\pi : S^2 \to \mathbb{R}P^2$ be the standard covering projection. It is regular and its group of covering transformations is $G = \{\text{id}, a_2\} \cong \mathbb{Z}_2$, where $a_2$ is as in Problem 5, so that $G$ acts without fixed points and $\mathbb{R}P^2 = S^2/G$. Since $G$ is finite,

$$
\pi^* : H^*_\text{deR}(\mathbb{R}P^2) \to H^*_\text{deR}(S^2)^G \equiv \{[\alpha] \in H^*_\text{deR}(S^2) : g^*[\alpha] = [\alpha] \forall g \in G\} = \{[\bar{\alpha}] \in H^*_\text{deR}(S^2) : a_2^*[\bar{\alpha}] = [\bar{\alpha}]\}
$$

is an isomorphism. Thus,

$$
H^0_\text{deR}(\mathbb{R}P^2) \cong H^0_\text{deR}(S^2)^G = \{f \in C^\infty(S^2) : f \text{ is const, } f \circ a_2 = f\} \approx \mathbb{R};
$$

$$
H^1_\text{deR}(\mathbb{R}P^2) \cong H^1_\text{deR}(S^2)^G \approx 0^G = 0.
$$

Finally, since $H^2_\text{deR}(S^2) \cong \mathbb{R}$ and $a_2 : S^2 \to S^2$ is orientation-reversing,

$$
a_2^* : H^2_\text{deR}(S^2) \to H^2_\text{deR}(S^2)
$$

is multiplication by a negative number (actually $-1$). Thus, $a_2^*$ does not fixed any nonzero element of $H^2_\text{deR}(S^2)$ and

$$
H^2_\text{deR}(\mathbb{R}P^2) \cong H^2_\text{deR}(S^2)^G = 0.
$$

(b) We begin by computing the de Rham cohomology of the complement $U$ in $\mathbb{R}P^2$ of a point or a small closed ball. Let $V$ be a slightly larger open ball. Since $U$ is a connected non-compact two-manifold,

$$
H^0_\text{deR}(U) \cong \mathbb{R}, \quad H^2_\text{deR}(U) = 0.
$$

Furthermore,

$$
H^p_\text{deR}(V) \cong \begin{cases} \mathbb{R}, & \text{if } p=0; \\ 0, & \text{otherwise}; \end{cases} \quad H^p_\text{deR}(\mathbb{R}P^2) \cong \begin{cases} \mathbb{R}, & \text{if } p=0; \\ 0, & \text{otherwise}. \end{cases}
$$

Since $U \cap V$ is an annulus and thus homotopy equivalent to $S^1$,

$$
H^p_\text{deR}(U \cap V) \cong \begin{cases} \mathbb{R}, & \text{if } p=0,1; \\ 0, & \text{otherwise}. \end{cases}
$$

By Mayer-Vietoris, we then have an exact sequence

$$
H^1_\text{deR}(\mathbb{R}P^2) \to H^1_\text{deR}(U) \oplus H^1_\text{deR}(V) \xrightarrow{\partial_1} H^1_\text{deR}(U \cap V) \to H^2_\text{deR}(\mathbb{R}P^2).
$$

Plugging in for the known groups, we obtain an exact sequence

$$
0 \to H^1_\text{deR}(U) \oplus 0 \xrightarrow{\partial_1} \mathbb{R} \to 0 \quad \implies \quad H^p_\text{deR}(U) \cong \begin{cases} \mathbb{R}, & \text{if } p=0,1; \\ 0, & \text{otherwise}. \end{cases}
$$
The 2-manifold $\mathbb{R}P^2 \# \mathbb{R}P^2$ is obtained by joining two copies of $U$, which we will call $U_1$ and $U_2$, along boundary annuli, so that

$$H^p_{\text{deR}}(U_1 \cap U_2) \cong H^p_{\text{deR}}(S^1) \cong \begin{cases} \mathbb{R}, & \text{if } p = 0, 1; \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, since $\mathbb{R}P^2 \# \mathbb{R}P^2$ is connected,

$$H^0_{\text{deR}}(\mathbb{R}P^2 \# \mathbb{R}P^2) \cong \mathbb{R}.$$

By Mayer-Vietoris, we obtain

$$
\begin{align*}
0 \longrightarrow H^0_{\text{deR}}(\mathbb{R}P^2 \# \mathbb{R}P^2) & \longrightarrow H^0_{\text{deR}}(U_1) \oplus H^0_{\text{deR}}(U_2) \longrightarrow H^0_{\text{deR}}(U_1 \cap U_2) \\
\delta_0 \longrightarrow H^1_{\text{deR}}(\mathbb{R}P^2 \# \mathbb{R}P^2) & \longrightarrow H^1_{\text{deR}}(U_1) \oplus H^1_{\text{deR}}(U_2) \longrightarrow H^1_{\text{deR}}(U_1 \cap U_2) \\
\delta_1 \longrightarrow H^2_{\text{deR}}(\mathbb{R}P^2 \# \mathbb{R}P^2) & \longrightarrow H^2_{\text{deR}}(U_1) \oplus H^2_{\text{deR}}(U_2).
\end{align*}
$$

Plugging in for known groups, we obtain

$$
\begin{align*}
0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} & \longrightarrow \mathbb{R} \\
\delta_0 \longrightarrow H^1_{\text{deR}}(\mathbb{R}P^2 \# \mathbb{R}P^2) & \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow H^2_{\text{deR}}(\mathbb{R}P^2 \# \mathbb{R}P^2) \longrightarrow 0.
\end{align*}
$$

By exactness of the sequence, the homomorphism $\delta_0$ is trivial, and the above sequence reduces to

$$
\begin{align*}
0 \longrightarrow H^1_{\text{deR}}(\mathbb{R}P^2 \# \mathbb{R}P^2) & \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \\
\delta_1 \longrightarrow H^2_{\text{deR}}(\mathbb{R}P^2 \# \mathbb{R}P^2) & \longrightarrow 0.
\end{align*}
$$

The restriction of the homomorphism $g_1$ to each component of the domain is the restriction homomorphism

$$g_1: H^1_{\text{deR}}(U) \longrightarrow H^1_{\text{deR}}(U \cap V)$$

of the previous paragraph, which is nontrivial. Thus, in this case $g_1$ must be nontrivial and $\delta_1$ trivial. Therefore, we obtain exact sequences

$$
\begin{align*}
0 \longrightarrow H^1_{\text{deR}}(\mathbb{R}P^2 \# \mathbb{R}P^2) & \longrightarrow \mathbb{R} \longrightarrow 0 \\
0 \longrightarrow H^2_{\text{deR}}(\mathbb{R}P^2 \# \mathbb{R}P^2) & \longrightarrow 0.
\end{align*}
$$

It follows that

$$H^1_{\text{deR}}(\mathbb{R}P^2 \# \mathbb{R}P^2) \cong \mathbb{R} \quad \text{and} \quad H^0_{\text{deR}}(\mathbb{R}P^2 \# \mathbb{R}P^2) = 0.$$

Alternatively, we can observe that $\mathbb{R}P^2 \# \mathbb{R}P^2$ is not orientable and therefore $H^0_{\text{deR}}(\mathbb{R}P^2 \# \mathbb{R}P^2) = 0$.

Here is another solution. Since $\mathbb{R}P^2 \# \mathbb{R}P^2$ is a connected and non-orientable 2-manifold (being the connect-sum of such manifolds),

$$H^0_{\text{deR}}(\mathbb{R}P^2 \# \mathbb{R}P^2) \cong \mathbb{R}, \quad H^2_{\text{deR}}(\mathbb{R}P^2 \# \mathbb{R}P^2) = 0.$$

On the other hand, by MAT 530, Hurewicz Theorem, and de Rham Theorem,

$$\pi_1(\mathbb{R}P^2 \# \mathbb{R}P^2) = \langle a, b \mid a^2b^2 \rangle \quad \implies \quad H_1(\mathbb{R}P^2 \# \mathbb{R}P^2; \mathbb{Z}) = \text{Abel}(\pi_1(\mathbb{R}P^2 \# \mathbb{R}P^2)) = \mathbb{Z} \oplus \mathbb{Z}_2$$

$$\implies \quad H_1(\mathbb{R}P^2 \# \mathbb{R}P^2; \mathbb{R}) \approx H_1(\mathbb{R}P^2 \# \mathbb{R}P^2; \mathbb{Z}) \otimes \mathbb{R} \approx \mathbb{R}$$

$$\implies \quad H^1_{\text{deR}}(\mathbb{R}P^2 \# \mathbb{R}P^2) \cong H^1(\mathbb{R}P^2 \# \mathbb{R}P^2; \mathbb{R}) \cong (H_1(\mathbb{R}P^2 \# \mathbb{R}P^2; \mathbb{R}))^* \cong \mathbb{R}.$$
Yet another approach is to observe that $\mathbb{R}P^2 \# \mathbb{R}P^2$ is homeomorphic to the Klein bottle and therefore by the de Rham Theorem

$$H^*_\text{deR}(\mathbb{R}P^2 \# \mathbb{R}P^2) \approx H^*_\text{deR}(K).$$

On the other hand, the orientable double cover of $K$ is the 2-torus and we can compute its cohomology as in Problem 5 on PS7.

**Part III (choose 1 problem from 7 and 8)**

7. (a) Show that the normal bundle of $S^n$ in $\mathbb{R}^{n+1}$ is trivial.

(b) Show that $S^3 \times S^4$ can be embedded into $\mathbb{R}^8$.

(c) Let $n_1, \ldots, n_k$ be nonnegative integers and $N$ their sum. Show that

$$S^{n_1} \times S^{n_2} \times \ldots \times S^{n_k}$$

can be embedded into $\mathbb{R}^{N+1}$.

(a) Since $\mathbb{R}^{n+1}$ and $S^n$ are orientable, the normal bundle of $S^n$ in $\mathbb{R}^{n+1}$ is orientable; see Problem 4a on PS6. Since an orientable line bundle is necessarily trivial, we conclude that the normal bundle of $S^n$ in $\mathbb{R}^{n+1}$ is trivial.

Alternatively, the normal bundle of $S^n$ in $\mathbb{R}^{n+1}$ is isomorphic to the bundle of vectors normal to $S^n$:

$$\mathcal{N} \equiv \{(x, v) : x \in S^n, v \in T_x \mathbb{R}^{n+1}, \langle v, w \rangle = 0 \forall w \in T_x S^n\} \rightarrow S^n.$$

It consists of radial directions along $S^n$. In particular, $\mathcal{N}$ admits a nowhere-vanishing section, $\partial/\partial r$. Since $\mathcal{N}$ is a line bundle, it follows that $\mathcal{N}$ is trivial.

(b) The normal bundle of

$$S^3 = S^3 \times 0 \subset \mathbb{R}^4 \times 0 \subset \mathbb{R}^4 \times \mathbb{R}^4 = \mathbb{R}^8$$

in $\mathbb{R}^8$ is the direct sum of the normal bundle of $S^3$ in $\mathbb{R}^4$ with the normal bundle of $\mathbb{R}^4$ in $\mathbb{R}^8$:

$$\mathcal{N}_{\mathbb{R}^8} S^3 = \mathcal{N}_{\mathbb{R}^4} S^3 \oplus \mathcal{N}_{\mathbb{R}^4} \mathbb{R}^4|_{S^3} \approx (S^3 \times \mathbb{R}) \oplus (S^3 \times \mathbb{R}^4) \approx S^3 \times \mathbb{R}^5.$$

On the other hand, a (tubular) neighborhood of $S^3$ in $\mathbb{R}^8$ is diffeomorphic to $\mathcal{N}_{\mathbb{R}^8} S^3$. Thus, this neighborhood contains a submanifold diffeomorphic to the sphere bundle of $\mathcal{N}_{\mathbb{R}^8} S^3$:

$$S(\mathcal{N}_{\mathbb{R}^8} S^3) \equiv \{v \in \mathcal{N}_{\mathbb{R}^8} S^3 : |v| = 1\} \approx S(S^3 \times \mathbb{R}^5) \approx S^3 \times S^4.$$

(c) For each $i=1, \ldots, k$, let

$$N_i = n_1 + \ldots + n_i \quad \text{and} \quad X_i = S^{n_1} \times \ldots \times S^{n_i}.$$

The sphere $X_1$ is an embedded submanifold of $\mathbb{R}^{N_1+1}$ by definition. Suppose $2 \leq i \leq k$ and $X_{i-1}$ has been embedded into $\mathbb{R}^{N_{i-1}+1}$. Since $X_{i-1}$ is an orientable $N_{i-1}$-manifold (being a product of orientable
manifolds), the normal bundle of $X_{i-1}$ in the orientable manifold $\mathbb{R}^{N_{i-1}+1}$ is an orientable line bundle and therefore a trivial line. The normal bundle of 

$$X_{i-1} = X_{i-1} \times 0 \subset \mathbb{R}^{N_{i-1}+1} \times 0 \subset \mathbb{R}^{N_{i-1}+1} \times \mathbb{R}^n = \mathbb{R}^{N_{i+1}}$$

in $\mathbb{R}^{N_{i+1}}$ is the direct sum of the normal bundle of $X_{i-1}$ in $\mathbb{R}^{N_{i-1}+1}$ with the normal bundle of $\mathbb{R}^{N_{i-1}+1}$ in $\mathbb{R}^{N_{i+1}}$:

$$
\mathcal{N}_{\mathbb{R}^{N_{i+1}}} X_{i-1} = \mathcal{N}_{\mathbb{R}^{N_{i-1}+1}} X_{i-1} \oplus \mathcal{N}_{\mathbb{R}^{N_{i+1}}} \mathbb{R}^{N_{i-1}+1}|_{X_{i-1}} \\
\approx (X_{i-1} \times \mathbb{R}) \oplus (X_{i-1} \times \mathbb{R}^n) \approx X_{i-1} \times \mathbb{R}^{n+1}.
$$

On the other hand, a (tubular) neighborhood of $X_{i-1}$ in $\mathbb{R}^{N_{i+1}}$ is diffeomorphic to $\mathcal{N}_{\mathbb{R}^{N_{i+1}}} X_{i-1}$. Thus, this neighborhood contains a submanifold diffeomorphic to the sphere bundle of $\mathcal{N}_{\mathbb{R}^{N_{i+1}}} X_{i-1}$:

$$S(\mathcal{N}_{\mathbb{R}^{N_{i+1}}} X_{i-1}) \equiv \{v \in \mathcal{N}_{\mathbb{R}^{N_{i+1}}} X_{i-1} : |v| = 1\}$$

$$\approx S(X_{i-1} \times \mathbb{R}^{n+1}) \approx X_{i-1} \times S^{n+1} = X_i.$$

Thus, $X_i$ embeds into $\mathbb{R}^{N_{i+1}}$. By induction this implies the claim.

8. Let $M$ be a smooth Riemannian manifold.

(a) What is the symbol of the differential,

$$d_p : E^p(M) \longrightarrow E^{p+1}(M)?$$

Under what conditions is this operator elliptic?

(b) What is the symbol of the (formal) adjoint of the differential,

$$\delta_p : E^p(M) \longrightarrow E^{p-1}(M)?$$

Under what conditions is this operator elliptic?

(c) What is the symbol of the operator

$$d+\delta : E^*(M) \longrightarrow E^*(M)?$$

Under what conditions is this operator elliptic?

(a) Suppose $x \in M$, $\alpha \in T^*_x M$, $\beta \in E^p(M)$, and $f \in C^\infty(M)$ is such that $f(x) = 0$ and $df|_x = \alpha$. Since $d_p$ is a first-order differential operator, by definition

$$\{ \sigma_{d_p}(\alpha) \} (\beta(x)) \equiv d(f \cdot \beta)|_x = df|_x \wedge \beta|_x + (-1)^0 f|_x \cdot d\beta|_x = \alpha \wedge \beta(x) + 0.$$

Thus, the symbol $\sigma_{d_p}$ of $d_p$ is the bundle homomorphism over $M$ given by

$$\sigma_{d_p} : T^* M \longrightarrow \text{Hom}(\Lambda^p(T^* M), \Lambda^{p+1}(T^* M)), \quad \{ \sigma_{d_p}(\alpha) \}(\beta) = \alpha \wedge \beta \quad \forall \alpha \in T^*_x M, \beta \in \Lambda^p(T^*_x M), x \in M.$$

The operator $d_p$ is elliptic if the homomorphism

$$\sigma_{d_p}(\alpha) : \Lambda^p(T^*_x M) \longrightarrow \Lambda^{p+1}(T^*_x M)$$
is an isomorphism for all $\alpha \in T_x^* M$, $\alpha \neq 0$, and $x \in M$. If this is the case, then

$$\binom{n}{p} = \dim \Lambda^p(T_x^* M) = \dim \Lambda^{p+1}(T_x^* M) = \binom{n}{p+1} = \frac{n-p}{p+1} \cdot \binom{n}{p},$$

where $n$ is the dimension of $M$. This means that $p = (n-1)/2$, so that $n$ is odd. On the other hand, if $p \geq 1$, then the homomorphism

$$\sigma_{d_p}(\alpha) : \Lambda^p(T_x^* M) \longrightarrow \Lambda^{p+1}(T_x^* M), \quad \beta \longrightarrow \alpha \wedge \beta,$$

is not injective; its kernel contains any element of the form $\alpha \wedge \gamma$. Thus, $\sigma_{d_p}(\alpha)$ is not an isomorphism in this case. The remaining case is $p = 0$ and $n = 1$. The homomorphism

$$\sigma_{d_p}(\alpha) : \Lambda^0(T_x^* M) = \mathbb{R} \longrightarrow \Lambda^1(T_x^* M) \approx \mathbb{R}, \quad \beta \longrightarrow \alpha \wedge \beta,$$

is then an isomorphism if $\alpha \neq 0$. Thus, $d_p$ is elliptic if and only if $M$ is a one-dimensional manifold and $p = 0$.

(b) Since $\delta_p$ is the formal adjoint of $d_{p-1}$ with respect to the inner-product $\langle \cdot, \cdot \rangle$ on $E^*(M)$ induced via integration by the point-wise inner-product $\langle \cdot, \cdot \rangle$ on $\Lambda^*(T^* M)$ and $d_{p-1}$ is a differential operator of odd (first) order, $\sigma_{\delta_p}(\alpha)$ is the negative adjoint of $\sigma_{d_{p-1}}(\alpha)$ with respect to $\langle \cdot, \cdot \rangle$:

$$\sigma_{\delta_p}(\alpha) = -\{\sigma_{d_{p-1}}(\alpha)\}^* : \Lambda^p(T_x^* M) \longrightarrow \Lambda^{p-1}(T_x^* M) \quad \forall \, \alpha \in T_x^* M, \, x \in M.$$

Since the inner-product $\langle \cdot, \cdot \rangle$ is nondegenerate, the homomorphism

$$T_x M \longrightarrow T_x^* M, \quad v \longrightarrow \langle v, \cdot \rangle,$$

is an isomorphism. For each $\alpha \in T_x^* M$, denote by $v_\alpha \in T_x M$ its preimage under this isomorphism. Since $\sigma_{d_{p-1}}(\alpha)$ is the left-wedging with $\alpha$ and $\sigma_{\delta_p}(\alpha)$ is its negative adjoint with respect to $\langle \cdot, \cdot \rangle$, it follows that

$$\sigma_{\delta_p}(\alpha) = -\iota_v : \Lambda^p(T_x^* M) \longrightarrow \Lambda^{p-1}(T_x^* M) \quad \forall \, \alpha \in T_x^* M, \, x \in M,$$

where $\iota_v$ is the contraction with respect to $v_\alpha$; see Section 2.11 in Warner.

Since $\delta_p$ is the formal adjoint of $d_{p-1}$, $\delta_p$ is elliptic if and only if $d_{p-1}$ is elliptic. By part (a), this is the case if and only if $M$ is a one-dimensional manifold and $p = 1$.

(c) Since $d$ and $\delta$ are differential operators of the same order (first), the symbol of their sum (as long as it is defined) is the sum of their symbols. By parts (a) and (b), this means that

$$\sigma_{d+\delta} = \sigma_d + \sigma_\delta : T^* M \longrightarrow \text{Hom}(\Lambda^*(T^* M), \Lambda^*(T^* M)), \quad \beta \longrightarrow \alpha \wedge \beta - \iota_v \beta, \quad \forall \, \alpha \in T_x^* M, \, \beta \in \Lambda^*(T_x^* M), \, x \in M.$$

Since $d^2 = 0$ and $\delta$ is the adjoint of $d$, $\delta^2 = 0$. Therefore,

$$(d+\delta)^2 = d\delta + \delta d = \Delta : E^*(M) \longrightarrow E^*(M).$$
In order to compute these cohomology groups, we’ll break $C$ and the complement of $C$ into three standard coordinate patches: Another possibility is to split the normal bundle of $C$. It remains to compute $H_\gamma$ of $C$. Note that $V$ is the tautological line bundle. In fact, we can take a rather large tubular neighborhood of $C^1$: Since $\Delta$ is an elliptic operator (see Section 63.5 in Warner), $d+\delta$ must be an elliptic operator as well. The reason for this is that

$$\sigma_\Delta(\alpha) = \sigma_{(d+\delta)\circ(d+\delta)}(\alpha) = \{\sigma_{d+\delta}(\alpha)\} \circ \{\sigma_{d+\delta}(\alpha)\}: \Lambda^*(T^*M) \longrightarrow \Lambda^*(T^*M) \quad \forall \alpha \in T_x^*M, \ x \in M.$$

Therefore, if $\sigma_\Delta(\alpha)$ is an isomorphism, then so is $\sigma_{d+\delta}(\alpha)$.

**Bonus Problem**

*Determine the cohomology ring of $\mathbb{C}P^2$.*

Since $\mathbb{C}P^2$ is connected (being a quotient of $S^5$),

$$H^0_{\text{def}}(\mathbb{C}P^2) \approx \mathbb{R}.$$

Since $\mathbb{C}P^2$ is a complex 2-manifold, it is also an orientable real 4-manifold. Since $\mathbb{C}P^2$ is compact, by Poincare Duality

$$H^4_{\text{def}}(\mathbb{C}P^2) \approx \mathbb{R}.$$

It remains to compute $H^1_{\text{def}}(\mathbb{C}P^2)$, $H^2_{\text{def}}(\mathbb{C}P^2)$, and $H^3_{\text{def}}(\mathbb{C}P^2)$.

In order to compute these cohomology groups, we’ll break $\mathbb{C}P^2$ into pieces. One natural choice is to break it into three standard coordinate patches:

$$U_i \equiv \{[X_0, X_1, X_2] \in \mathbb{C}P^2 : X_i \neq 0\} \approx \mathbb{C}^2, \quad i = 0, 1, 2.$$

Another possibility is to split $\mathbb{C}P^2$ into a tubular neighborhood $V$ of

$$\mathbb{C}P^1 = \{[X_0, X_1, X_2] \in \mathbb{C}P^2 : X_2 = 0\}$$

and the complement of $\mathbb{C}P^1$ in $\mathbb{C}P^2$. The latter is precisely $U_2$, while the former is diffeomorphic to the normal bundle of $\mathbb{C}P^1$ in $\mathbb{C}P^2$. By Problem 6 on PS2, this normal bundle is isomorphic to $\gamma_1^*$, where $\gamma_1 \longrightarrow \mathbb{C}P^1$ is the tautological line bundle. In fact, we can take a rather large tubular neighborhood of $\mathbb{C}P^1$:

$$V \equiv \{[X_0, X_1, X_2] \in \mathbb{C}P^2 : (X_0, X_1) \neq 0\} \approx \mathbb{C}^2, \quad [X_0, X_1, \alpha(X_0, X_1)] \longrightarrow ([X_0, X_1], \alpha).$$

Note that $V = U_0 \cup U_1$. While we could determine the cohomology of $V$ via Mayer-Vietoris, this turns out to be unnecessary.

We will determine the cohomology of $\mathbb{C}P^2$ using the splitting $\mathbb{C}P^2 = U_2 \cup V$. First,

$$U_2 \cap V = \{[X_0, X_1, X_2] \in \mathbb{C}P^2 : (X_0, X_1) \neq 0, X_2 \neq 0\} \approx \mathbb{C}^2 - 0, \quad [X_0, X_1, X_2] \longrightarrow (X_0/X_2, X_1/X_2).$$

Thus, $U_2 \cap V$ is homotopy equivalent to $S^3 \subset \mathbb{C}^2$. Therefore, $U_2 \cap V$ is connected and

$$H^p_{\text{def}}(U_2 \cap V) \approx \begin{cases} \mathbb{R}, & \text{if } p = 0, 3; \\ 0, & \text{otherwise.} \end{cases}$$

\footnote{For the second equality to hold, the definition of symbol given in class and in Warner 6.28(3) should be multiplied by $1/k!$, where $k$ is the order of the operator.}
Since $V$ is diffeomorphic to a vector bundle over $\mathbb{C}P^1 \approx S^2$, $V$ is homotopy equivalent to $S^2$. In particular, $V$ is connected, simply-connected, and

$$H^p_{\text{deR}}(V) \approx \begin{cases} \mathbb{R}, & \text{if } p = 0, 2; \\ 0, & \text{otherwise.} \end{cases}$$

Finally, $U_2$ is diffeomorphic to $\mathbb{C}^2$. Therefore, $U_2$ is connected, simply-connected, and

$$H^p_{\text{deR}}(U_2) \approx \begin{cases} \mathbb{R}, & \text{if } p = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, by van Kampen Theorem, $\mathbb{C}P^2 = U_2 \cup V$ is also simply connected. By Hurewicz Theorem, de Rham Theorem, and Poincare Duality,

$$H_1(\mathbb{C}P^2; \mathbb{Z}) = \text{Abel}(\pi_1(\mathbb{C}P^2)) = 0, \quad H_1(\mathbb{C}P^2; \mathbb{R}) \approx H_1(\mathbb{C}P^2; \mathbb{Z}) \otimes \mathbb{R} = 0,$$

$$H^1_{\text{deR}}(\mathbb{C}P^2) \approx H^1(\mathbb{C}P^2; \mathbb{R}) = (H_1(\mathbb{C}P^2; \mathbb{R}))^* = 0, \quad H^3_{\text{deR}}(\mathbb{C}P^2) \approx (H^1_{\text{deR}}(\mathbb{C}P^2))^* = 0.$$ 

By Mayer-Vietoris, we have an exact sequence

$$H^1_{\text{deR}}(U_2 \cap V) \longrightarrow H^2_{\text{deR}}(\mathbb{C}P^2) \longrightarrow H^2_{\text{deR}}(U_2) \oplus H^2_{\text{deR}}(V) \longrightarrow H^2_{\text{deR}}(U_2 \cap V).$$

Plugging in for the known groups, we obtain an exact sequence

$$0 \longrightarrow H^2_{\text{deR}}(\mathbb{C}P^2) \longrightarrow \mathbb{R} \longrightarrow 0 \quad \implies \quad H^2_{\text{deR}}(\mathbb{C}P^2) \approx \mathbb{R}.$$ 

Finally, by Poincare Duality, the pairing

$$H^2_{\text{deR}}(\mathbb{C}P^2) \otimes H^2_{\text{deR}}(\mathbb{C}P^2) \longrightarrow \mathbb{R}, \quad [\alpha] \otimes [\beta] \longrightarrow \int_{\mathbb{C}P^2} \alpha \wedge \beta,$$ 

is nondegenerate. Therefore, we have an isomorphism of graded rings

$$H^*_{\text{deR}}(\mathbb{C}P^2) \approx \mathbb{R}[u]/u^3,$$

where the degree of $u$ is defined to be 2.

Note: We can compute $H^1_{\text{deR}}(\mathbb{C}P^2)$ and $H^3_{\text{deR}}(\mathbb{C}P^2)$ from MV as well. The natural generator $u$ for $H^0_{\text{deR}}(\mathbb{C}P^2)$ is the first chern class of the line bundle $\gamma_2^* \longrightarrow \mathbb{C}P^2$. You should now be able to guess what the de Rham cohomology of $\mathbb{C}P^n$ is and to prove your guess inductively.