MAT 530: Topology&Geometry, I
Fall 2005

Problem Set 7

Solution to Problem p271, #10

Uniqueness-of-Completion Theorem: Let (X,d) be metric space. Suppose (Y,D) and (Y',D') are complete metric space such that

$$X \subset Y, Y', \qquad \mathrm{Cl}_Y X = Y, \qquad \mathrm{Cl}_{Y'} X = Y', \qquad D|_{X\times X} = d, \quad \textit{and} \quad D'|_{X\times X} = d.$$

Show that there exists an isometry $f\colon (Y,D) \longrightarrow (Y',D')$ that restricts to the identity on X.

Given $y \in Y$, we define $f(y) \in Y'$ as follows. Since $\mathrm{Cl}_Y X = Y$ and Y is first-countable, there exists a sequence of points

$$x_1, x_2, \ldots \in X \qquad \text{s.t.} \qquad \lim_{n\longrightarrow\infty} x_n = y \in Y.$$

Thus, this sequence must be Cauchy in (Y,D). Since D agrees with d over X, it is also Cauchy in (X,d). Since D' agrees with d over X, the sequence

$$x_1, x_2, \ldots \in X \subset Y'$$

is Cauchy in (Y',D'). Since (Y',D') is complete, this sequence must converge to a limit point $y' \in Y'$. We then set

$$f(y) = y' \in Y'.$$

First, we need to show that $f(y)$ is well-defined, i.e. that it depends only on y, and not on the choice of the sequence as above. Suppose we have another sequence:

$$\tilde{x}_1, \tilde{x}_2, \ldots \in X \qquad \text{s.t.} \qquad \lim_{n\longrightarrow\infty} \tilde{x}'_n = y \in Y.$$

Let $\tilde{y}'$ be the limit of this sequence in (Y',D'). We will show that $\tilde{y}' = y'$. We define a third sequence of points in X by combining the two sequences:

$$z_n = \begin{cases} x_k, & \text{if } n = 2k-1; \\ \tilde{x}_k, & \text{if } n = 2k. \end{cases}$$

This sequence still converges to y in Y. Thus, it is still Cauchy in (Y',D') and thus converges to a point $z' \in Y'$. Since every subsequence of $z_1, z_2, \ldots$ must also converge to z' in Y, we conclude that $y' = z' = \tilde{y}'$.

If $y \in X$, we can take the corresponding sequence to be given by $x_n = y$ for all n. This sequence converges to y in Y'. Thus, $f(x) = x$ for all $x \in X$.

We next show that $f\colon (Y,D)\longrightarrow (Y',D')$ is an isometric embedding. Suppose

$$y,\tilde{y}\in Y, \qquad x_1,x_2,\ldots\in X \qquad \tilde{x}_1,\tilde{x}_2,\ldots\in X, \qquad \lim_{n\longrightarrow\infty} x_n = y\in Y, \quad \text{and} \quad \lim_{n\longrightarrow\infty}\tilde{x}_n=\tilde{y}\in Y.$$

It then follows that

$$\begin{aligned} D(y,\tilde{y}) &= \lim_{n\longrightarrow\infty} D(x_n,\tilde{x}_n) = \lim_{n\longrightarrow\infty} d(x_n,\tilde{x}_n) \qquad \text{and} \\ D'(f(y),f(\tilde{y})) &= \lim_{n\longrightarrow\infty} D'(x_n,\tilde{x}_n) = \lim_{n\longrightarrow\infty} d(x_n,\tilde{x}_n). \end{aligned}$$

Thus, $D'(f(y),f(\tilde{y}))=D(y,\tilde{y})$.

It remains to show that f is surjective. Suppose $y'\in Y'$. Since $\mathrm{Cl}_{Y'}X=Y'$ and Y' is first-countable, there exists a sequence of points

$$x_1,x_2,\ldots\in X \qquad \text{s.t.} \qquad \lim_{n\longrightarrow\infty} x_n = y'\in Y'.$$

Thus, this sequence must be Cauchy in (Y',D'). Since D' agrees with d over X, it is also Cauchy in (X,d). Since D agrees with d over X, the sequence

$$x_1,x_2,\ldots\in X\subset Y$$

is Cauchy in (Y,D). Since (Y,D) is complete, this sequence must converge to a limit point $y\in Y$. Then, by definition, $f(y)=y'$.

Solution to Problem p274, #2

Show that there is a continuous surjective map $f\colon\mathbb{R}\longrightarrow\mathbb{R}^n$.

Below we use a surjective map $f:I\longrightarrow I^2$ to construct a surjective map $g:\mathbb{R}\longrightarrow\mathbb{R}^2$. There is more than one way of doing this. Once this is done, we can obtain a surjective map $\mathbb{R}\longrightarrow\mathbb{R}^{2^n}$ by compositing f with $f\times f$, then with $f\times f\times f\times f$, and so on, n times. We then compose the resulting map with a projection $\mathbb{R}^{2^n}\longrightarrow\mathbb{R}^n$.

Let $n\in\mathbb{Z}^+$. Since the interval $[2n-1,2n]$ is homeomorphic to $[0,1]$ and the square $[-n,n]^2$ is homeomorphic to I^2, by the Piano Curve Theorem there exists a continuous surjective map

$$f_n\colon [2n-1,2n]\longrightarrow [-n,n]^2.$$

Since all intervals $[2n–1,2n]$ are disjoint, putting these maps together we obtain a continuous surjective map

$$\tilde{f}\colon \bigcup_{n\in\mathbb{Z}^+}[2n-1,2n]\longrightarrow \bigcup_{n\in\mathbb{Z}^+}[-n,n]^2=\mathbb{R}^2.$$

Since $\bigcup_{n\in\mathbb{Z}^+}[2n-1,2n]$ is closed in $\mathbb{R}$ and $\mathbb{R}$ is normal, by the Tietze Extension Theorem (applied to each component of $\tilde{f}$) $\tilde{f}$ extends to a continuous map $g\colon\mathbb{R}\longrightarrow\mathbb{R}^2$. Since it extends $\tilde{f}$, g must also be surjective.

Remark: In this case we do not need to use the Tietze Extension Theorem. We can simply extend $\tilde{f}$ linearly over each interval $[2n, 2n+1]$ and map all of $(-\infty, 1]$ to $\tilde{f}(1)$.

Here is a variation on this construction. Let $\{I_n^2 : n \in \mathbb{Z}^+\}$ be the set of all unit squares in $\mathbb{R}^2$ with vertices on $\mathbb{Z}^2$, ordered in some way by $\mathbb{Z}^+$. For each $n \in \mathbb{Z}^+$, let

$$f_n \colon [2n-1, 2n] \longrightarrow I_n^2$$

be a continuous surjective map. Since all intervals $[2n-1, 2n]$ are disjoint, putting these maps together we obtain a continuous surjective map

$$\tilde{f} \colon \bigcup_{n \in \mathbb{Z}^+} [2n-1, 2n] \longrightarrow \bigcup_{n \in \mathbb{Z}^+} I_n^2 = \mathbb{R}^2.$$

As above, this map extends to a continuous surjective map on all of $\mathbb{R}$.

Here is a different construction. Let $f \colon I \longrightarrow I^2$ be a continuous surjective map. We can assume that

$$f(0) = 0 \times 0, \qquad f(1) = 1 \times 0, \quad \text{and} \quad f^{-1}\big(0 \times 0\big) = \{0\}.$$

The map constructed in the proof of Theorem 44.1 necessarily satisfies the first two conditions and with some care can be made to satisfy the third (e.g. if at each step, the map gets stretched uniformly). Since $([1,2], 2)$ is homeomorphic to $([0,1], 0)$, there also exists a continuous surjective map $g \colon [1,2] \longrightarrow I^2$ such that

$$g(2) = 0 \times 0, \qquad g(1) = 1 \times 0, \quad \text{and} \quad g^{-1}\big(0 \times 0\big) = \{2\}.$$

Let $p, q \colon I^2 \longrightarrow S^2$ be homeomorphisms onto the top and bottom hemispheres such that

$$p(0 \times 0) = q(0 \times 0) \equiv a \qquad \text{and} \qquad p(1 \times 0) = q(1 \times 0).$$

Then, the map

$$p \circ f \colon [0,1] \longrightarrow S^2 \qquad \text{and} \qquad q \circ g \colon [1,2] \longrightarrow S^2$$

are continuous, the first surjective onto the upper hemisphere, and the second on the lower. Furthermore,

$$\{p \circ f\}(1) = p(1 \times 0) = q(1 \times 0) = \{q \circ g\}(1), \qquad \{p \circ f\}^{-1}(a) = 0, \quad \text{and} \quad \{q \circ g\}^{-1}(a) = \{2\}.$$

Thus, $p \circ f$ and $q \circ g$ patch together to produce a continuous surjective map

$$h \colon [0,2] \longrightarrow S^2 \qquad \text{s.t.} \qquad h^{-1}(a) = \{0, 2\}.$$

The restriction of this map to $(0,2)$ surjects onto $S^2 - \{a\}$. Since $(0,2)$ is homeomorphic to $\mathbb{R}$ and $S^2 - \{a\}$ is homeomorphic to $\mathbb{R}^2$, we are done.