MAT 530: Topology&Geometry, I
Fall 2005

Problem Set 4

Solution to Problem p67, #7

Let $J$ be a well-ordered set. A subset $J_0$ of $J$ is **inductive** if for every $\alpha \in J$

$$S_\alpha \subset J_0 \implies \alpha \in J_0.$$ 

**Proposition** (Principle of Transfinite Induction) If $J$ is a well-ordered set and $J_0 \subset J$ is inductive, then $J_0 = J$.

Recall that if $J$ is an ordered set, $S_\alpha$ is the section of $J$ by $\alpha$:

$$S_\alpha = \{ x \in J : x < \alpha \}.$$

Suppose $J$ is a well-ordered set, $J_0 \subset J$ is inductive, and $J_0 \neq J$, i.e. the set $J - J_0$ is nonempty. Since $J$ is well-ordered, every nonempty subset of $J$ has a minimal element. Let

$$\alpha = \min(J - J_0) \in J - J_0 \implies \alpha \notin J_0.$$  

(1)

Since $\alpha$ is the minimal element of $J - J_0$, $x \in J_0$ for all $x < \alpha$, i.e. $S_\alpha \subset J_0$. Since $J_0$ is inductive, it follows that $\alpha \in J_0$, contrary to (1).

Solution to Problem p235, #1

Let $X$ be a space. Let $D$ a collection of subsets of $X$ that is maximal with respect to the finite-intersection property.

(a) Show that $x \in \bar{D}$ for every $D \in D$ if and only if every neighborhood of $x$ belongs to $D$. Which implication uses the maximality of $D$?

(b) Show that if $D \in D$ and $D \subset A \subset X$, then $A \in D$.

(c) Show that if $X$ satisfies the T1-axiom, $\bigcap_{D \in D} \bar{D}$ contains at most one point.

(a) By Theorem 17.5, $x \in \bar{D}$ for every $D \in D$ if and only if every neighborhood of $x$ intersects every element $D$ of $D$. Since $D$ has the finite-intersection property and is maximal with respect to this property, by (b) of Lemma 37.2 the latter is the case if and only if every neighborhood of $x$ belongs to $D$. This proves (a). The maximality of $D$ is used to show that if $x \in \bar{D}$ for every $D \in D$, then every neighborhood of $x$ belongs to $D$ (i.e. the only if part of the claim).

(b) Since $D \subset A$ and $D$ intersects every element of $D$, so does $A$. The desired conclusion follows from (b) of Lemma 37.2.
(c) Suppose \( X \) is Hausdorff and \( x \) and \( y \) are two distinct points in \( \bigcap_{D \in D} \bar{D} \). Since \( x \neq y \), there exist disjoint neighborhoods \( U \) and \( V \) of \( x \) and \( y \), respectively. Since \( x \in \bar{D} \) for every \( D \in D \), by part (a) \( U \in D \). Since \( y \in \bar{D} \) for every \( D \in D \), by part (a) \( V \in D \). However, \( U \cap V = \emptyset \), which is impossible since \( D \) has the finite intersection property.

The T1 assumption is not sufficient here. Here is a counterexample. Let \( X \) be an infinite set endowed with the finite complement topology. Let \( A \) be the collection of all nonempty open sets in \( X \), i.e. \( A \) is the collection of all sets whose complement is finite. Since \( X \) is infinite, the intersection of any finite collection of elements of \( A \) is nonempty; in fact, it is another element of \( A \). Thus, \( A \) has the finite intersection property. By Lemma 37.1, there exists a collection \( D \) of subsets of \( X \) such that \( D \) has the finite intersection property, is maximal with respect to this property, and contains \( A \). Since \( D \) has the finite intersection property and contains \( A \), every elements \( D \) of \( D \) must intersect every element of \( A \), i.e. every nonempty open set in \( X \). Thus, \( \bar{D} = X \) for all \( D \in D \). It follows that \( \bigcap_{D \in D} \bar{D} = X \).

Remark: The first edition of the book actually had the correct statement. This error is also listed in a publicly available list of corrections to the second edition. Nevertheless, sometimes it is good to see whether any assumptions in a statement can be weakened.

Solution to Problem p236, #5

Prove Tychonoff’s Theorem using the open-set definition of compactness and the Tube Lemma.

**Theorem:** If \( X_j \) is a compact topological space for every \( j \in J \), then the space \( X = \prod_{j \in J} X_j \) is compact in the product topology.

**Tube Lemma:** Suppose \( X \) and \( Y \) are topological spaces, \( X \) is compact, and \( A \) is a collection of standard basis elements for the topology of \( X \times Y \). If no finite subcollection of \( A \) covers \( X \times Y \), then there exists \( x \in X \) such that no finite subcollection of \( A \) covers \( x \times Y \).

**Proof of Theorem:** For each \( j \in J \), let \( \pi_j : X \rightarrow X_j \) be the projection map. Choose a well-ordering on \( J \) so that \( J \) has a maximal element, \( j_{\text{max}} \). Denote the minimal element by \( j_{\text{min}} \).

(a) Suppose \( k \in J \), \( k \neq j_{\text{min}} \), and \( p_i \in X_i \) is a point for each \( i < k \). For each \( j < k \), let

\[
Y_j = \left\{ (x_i)_{i \in J} : x_i = p_i \quad \forall \; i \leq j \right\}.
\]

Then, \( j < j' < k \implies Y_j \supset Y_{j'} \). Let

\[
Z_k = \bigcap_{j < k} Y_j = \left\{ (x_i)_{i \in J} : x_i = p_i \quad \forall \; i < k \right\}.
\]

Show that if \( A \) is a finite collection of basis elements that covers \( Z_k \), then \( A \) covers \( Y_j \) for some \( j < k \).

(b) Suppose \( A \) is a collection of standard basis elements for \( X \) such that no finite subcollection of \( A \) covers \( X \). Show that there are points \( p_i \in X_i \) for each \( i \in J \) such that every corresponding space \( Y_j \) defined in (a) cannot be covered by a finite subcollection of \( A \). Thus,

\[
Y_{j_{\text{max}}} = \left\{ (p_i)_{i \in J} \right\}
\]
is a one-point set which is not contained in any of the elements of \( A \). Thus, \( A \) does not cover \( X \).

(a) If \( k \) has an immediate predecessor, \( j^* \), in \( J \), then

\[
Z_k = \bigcap_{j<k} Y_j = Y_{j^*},
\]

and \( A \) covers \( Y_{j^*} \). Suppose \( k \) does not have an immediate predecessor and thus the set

\[
\{i \in J : i < k\}
\]

is infinite. For each \( \mathcal{U} \in \mathcal{A} \), let

\[
J_{\mathcal{U}} = \{i \in J : i < k, \pi_i(\mathcal{U}) \neq X_i\}.
\]

Since \( \mathcal{U} \) is a basis element in the product topology, the set \( J_{\mathcal{U}} \) is finite for every \( \mathcal{U} \in \mathcal{A} \). Since \( \mathcal{A} \) is finite, \( \bigcup_{\mathcal{U} \in \mathcal{A}} J_{\mathcal{U}} \) is finite. Thus, there exists

\[
j^* \in J \quad \text{s.t.} \quad j^* < k \quad \text{and} \quad i < j^* \quad \forall i \in \bigcup_{\mathcal{U} \in \mathcal{A}} J_{\mathcal{U}}.
\]

Every element \( \mathcal{U} \) of \( \mathcal{A} \) must then be of the form

\[
\mathcal{U} = \mathcal{U}_- \times \prod_{j^* \leq j < k} X_j \times \mathcal{U}_+ \subset \prod_{j < j^*} X_j \times \prod_{j^* < j < k} X_j \times \prod_{k \leq j} X_j
\]

for some open subsets \( \mathcal{U}_- \) and \( \mathcal{U}_+ \) of \( \prod_{j < j^*} X_j \) and \( \prod_{k \leq j} X_j \), dependent on \( \mathcal{U} \). Since the sets \( \mathcal{U} \in \mathcal{A} \) cover \( Z_k \), the sets \( \mathcal{U}_- \times \mathcal{U}_+ \) cover

\[
(p_i)_{i < j^*} \times \prod_{k \leq j} X_j,
\]

It follows that the sets

\[
\mathcal{U} = \mathcal{U}_- \times \prod_{j^* \leq j < k} X_j \times \mathcal{U}_+
\]

cover

\[
(p_i)_{i < j^*} \times p_{j^*} \times \prod_{j^* < j < k} X_j \times \mathcal{U}_+ = Y_{j^*}.
\]

(b) Since the space \( X_{j_{\text{min}}} \) is compact by assumption and the space

\[
X_{j_{\text{min}}} \times \prod_{j_{\text{min}} < j} X_j
\]

cannot be covered by finitely many elements of \( \mathcal{A} \), by the Tube Lemma there exists \( p_{j_{\text{min}}} \in X_{j_{\text{min}}} \) such that

\[
p_{j_{\text{min}}} \times \prod_{j_{\text{min}} < j} X_j
\]
cannot be covered by finitely many elements of $A$. Suppose $k \in J$, $k \in j_{\min}$, and we have chosen $p_i \in X_i$ for each $j < k$ so that the corresponding slices $Y_j$, with $j < k$, of part (a) cannot be covered by finitely many elements of $A$. By part (a), the corresponding space

$$Z_k = (p_i)_{i < k} \times X_k \times \prod_{k < i} X_i$$

cannot be covered by finitely many elements of $A$. Since the space $X_k$ is compact, by the Tube Lemma there exists $p_k \in X_k$ such that

$$Z_k = (p_i)_{i < k} \times p_k \times \prod_{k < i} X_i = (p_i)_{i \leq k} \times \prod_{k < i} X_i$$

cannot be covered by finitely many elements of $A$, as needed.

**Remark 1:** By part (b), if $A$ is an open cover of $X$ by basis elements, then $A$ has a finite subcover. This implies that $X$ is compact.

**Remark 2:** The argument in part (b) above, as suggested by the book, is actually a little problematic. Here is a fix. Let $S$ be the set of all subsets $I$ of $J$ such that if $j \in I$, $i \in J$, and $i < j$, then $i \in I$. Let $D$ be the collection of all elements $(p_i)_{i \in I}$ of

$$\bigcup_{I \in S} \prod_{j \in I} X_j$$

such that the corresponding slices $Y_j$, with $j \in I$, defined in part (a) cannot be covered by finitely many elements of $A$. We define a partial ordering on $D$ by

$$(p_i)_{i \in I} \prec (p'_i)_{i \in I'}$$ if $I \subseteq I'$ and $p_i = p'_i \ \forall i \in I$.

By Zorn’s Lemma (or the Maximum Principle), there exists a maximal simply ordered subset $D^*$ of $D$. Let $I^*$ be the union of all sets $I$ such that $(p_i)_{i \in I}$ is an element of $D^*$. Since $D^*$ is maximal, $(p_i)_{i \in I^*}$ is an element of $D^*$ and is the largest element of $D^*$. We claim that $I^* = J$. If not, take $k$ to be the smallest element of $J - I^*$ and proceed as above to choose $p_k \in X_k$. The element $(p_i)_{i \in I^* \cup \{k\}}$ of $D$ is larger than any element of $D^*$, contrary to the assumption that $D^*$ is a maximal simply ordered subcollection of $D$. 