MAT 530: Topology&Geometry, I
Fall 2005

Problem Set 4

Solution to Problem p67, #7

Let J be a well-ordered set. A subset J_0 of J is **inductive** *if for every $\alpha \in J$*

$$S_\alpha \subset J_0 \qquad \Longrightarrow \qquad \alpha \in J_0.$$

Proposition *(Principle of Transfinite Induction) If J is a well-ordered set and $J_0 \subset J$ is inductive, then $J_0 = J$.*

Recall that if J is an ordered set, S_α is the section of J by α:

$$S_\alpha = \{x \in J : x < \alpha\}.$$

Suppose J is a well-ordered set, $J_0 \subset J$ is inductive, and $J_0 \neq J$, i.e. the set $J - J_0$ is nonempty. Since J is well-ordered, every nonempty subset of J has a minimal element. Let

$$\alpha = \min(J - J_0) \in J - J_0 \qquad \Longrightarrow \qquad \alpha \notin J_0. \tag{1}$$

Since α is the minimal element of $J - J_0$, $x \in J_0$ for all $x < \alpha$, i.e. $S_\alpha \subset J_0$. Since J_0 is inductive, it follows that $\alpha \in J_0$, contrary to (1).

Solution to Problem p235, #1

Let X be a space. Let $\mathcal{D}$ a collection of subsets of X that is maximal with respect to the finite-intersection property.
(a) Show that $x \in \bar{D}$ for every $D \in \mathcal{D}$ if and only if every neighborhood of x belongs to $\mathcal{D}$. Which implication uses the maximality of $\mathcal{D}$?
(b) Show that if $D \in \mathcal{D}$ and $D \subset A \subset X$, then $A \in \mathcal{D}$.
(c) Show that if X satisfies the T1-axiom, $\bigcap_{D \in \mathcal{D}} \bar{D}$ contains at most one point.

(a) By Theorem 17.5, $x \in \bar{D}$ for every $D \in \mathcal{D}$ if and only if every neighborhood of x intersects every element D of $\mathcal{D}$. Since $\mathcal{D}$ has the finite-intersection property and is maximal with respect to this property, by (b) of Lemma 37.2 the latter is the case if and only if every neighborhood of x belongs to $\mathcal{D}$. This proves (a). The maximality of $\mathcal{D}$ is used to show that if $x \in \bar{D}$ for every $D \in \mathcal{D}$, then every neighborhood of x belongs to $\mathcal{D}$ (i.e. the *only if* part of the claim).

(b) Since $D \subset A$ and D intersects every element of $\mathcal{D}$, so does A. The desired conclusion follows from (b) of Lemma 37.2.

(c) Suppose X is *Hausdorff* and x and y are two distinct points in $\bigcap_{D\in\mathcal{D}}\bar{D}$. Since $x\neq y$, there exist disjoint neighborhoods U and V of x and y, respectively. Since $x\in\bar{D}$ for every $\mathcal{D}\in\mathcal{D}$, by part (a) $U\in\mathcal{D}$. Since $y\in\bar{D}$ for every $\mathcal{D}\in\mathcal{D}$, by part (a) $V\in\mathcal{D}$. However, $U\cap V=\emptyset$, which is impossible since $\mathcal{D}$ has the finite intersection property.

The T1 assumption is not sufficient here. Here is a counterexample. Let X be an infinite set endowed with the finite complement topology. Let $\mathcal{A}$ be the collection of all nonempty open sets in X, i.e. $\mathcal{A}$ is the collection of all sets whose complement is finite. Since X is infinite, the intersection of any finite collection of elements of $\mathcal{A}$ is nonempty; in fact, it is another element of $\mathcal{A}$. Thus, $\mathcal{A}$ has the finite intersection property. By Lemma 37.1, there exists a collection $\mathcal{D}$ of subsets of X such that $\mathcal{D}$ has the finite intersection property, is maximal with respect to this property, and contains $\mathcal{A}$. Since $\mathcal{D}$ has the finite intersection property and contains $\mathcal{A}$, every elements D of $\mathcal{D}$ must intersect every element of $\mathcal{A}$, i.e. every nonempty open set in X. Thus, $\bar{D}=X$ for all $D\in\mathcal{D}$. It follows that $\bigcap_{D\in\mathcal{D}}\bar{D}=X$.

Remark: The first edition of the book actually had the correct statement. This error is also listed in a publicly available list of corrections to the second edition. Nevertheless, sometimes it is good to see whether any assumptions in a statement can be weakened.

Solution to Problem p236, #5

Prove Tychonoff's Theorem using the open-set definition of compactness and the Tube Lemma.
Theorem: *If X_j is a compact topological space for every $j\in J$, then the space $X=\prod_{j\in J}X_j$ is compact in the product topology.*
Tube Lemma: *Suppose X and Y are topological spaces, X is compact, and $\mathcal{A}$ is a collection of standard basis elements for the topology of $X\times Y$. If no finite subcollection of $\mathcal{A}$ covers $X\times Y$, then there exists $x\in X$ such that no finite subcollection of $\mathcal{A}$ covers $x\times Y$.*

Proof of Theorem: For each $j\in J$, let $\pi_j: X\longrightarrow X_j$ be the projection map. Choose a well-ordering on J so that J has a maximal element, $j_{\max}$. Denote the minimal element by $j_{\min}$.
(a) Suppose $k\in J$, $k\neq j_{\min}$, and $p_i\in X_i$ is a point for each $i<k$. For each $j<k$, let

$$Y_j=\big\{(x_i)_{i\in J}: x_i=p_i\ \forall\, i\le j\big\}.$$

Then, $j<j'<k \implies Y_j\supset Y_{j'}$. Let

$$Z_k=\bigcap_{j<k}Y_j=\big\{(x_i)_{i\in J}: x_i=p_i\ \forall\, i<k\big\}.$$

Show that if $\mathcal{A}$ is a finite collection of basis elements that covers Z_k, then $\mathcal{A}$ covers Y_j for some $j<k$.
(b) Suppose $\mathcal{A}$ is a collection of standard basis elements for X such that no finite subcollection of $\mathcal{A}$ covers X. Show that there are points $p_i\in X_i$ for each $i\in J$ such that every corresponding space Y_j defined in (a) cannot be covered by a finite subcollection of $\mathcal{A}$. Thus,

$$Y_{j_{\max}}=\big\{(p_i)_{i\in J}\big\}$$

is a one-point set which is not contained in any of the elements of $\mathcal{A}$. Thus, $\mathcal{A}$ does not cover X.

(a) If k has an immediate predecessor, j^*, in J, then

$$Z_k = \bigcap_{j<k} Y_j = Y_{j^*},$$

and $\mathcal{A}$ covers Y_{j^*}. Suppose k does not have an immediate predecessor and thus the set

$$\{i \in J : i < k\}$$

is infinite. For each $\mathcal{U} \in \mathcal{A}$, let

$$J_{\mathcal{U}} = \big\{ i \in J : i < k,\ \pi_i(\mathcal{U}) \neq X_i \big\}.$$

Since $\mathcal{U}$ is a basis element in the product topology, the set $J_{\mathcal{U}}$ is finite for every $\mathcal{U} \in \mathcal{A}$. Since $\mathcal{A}$ is finite, $\bigcup_{\mathcal{U}\in\mathcal{A}} J_{\mathcal{U}}$ is finite. Thus, there exists

$$j^* \in J \qquad \text{s.t.} \qquad j^* < k \quad \text{and} \quad i < j^* \quad \forall\, i \in \bigcup_{\mathcal{U}\in\mathcal{A}} J_{\mathcal{U}}.$$

Every element $\mathcal{U}$ of $\mathcal{A}$ must then be of the form

$$\mathcal{U} = \mathcal{U}_- \times \prod_{j^* \le j < k} X_j \times \mathcal{U}_+ \subset \prod_{j<j^*} X_j \times \prod_{j^* \le j < k} X_j \times \prod_{k \le j} X_j$$

for some open subsets $\mathcal{U}_-$ and $\mathcal{U}_+$ of $\prod_{j<j^*} X_j$ and $\prod_{k\le j} X_j$, dependent on $\mathcal{U}$. Since the sets $\mathcal{U} \in \mathcal{A}$ cover Z_k, the sets $\mathcal{U}_- \times \mathcal{U}_+$ cover

$$(p_i)_{i<j^*} \times \prod_{k \le j} X_j.$$

It follows that the sets

$$\mathcal{U} = \mathcal{U}_- \times \prod_{j^* \le j < k} X_j \times \mathcal{U}_+$$

cover

$$(p_i)_{i<j^*} \times p_{j^*} \times \prod_{j^* < j < k} X_j \times \mathcal{U}_+ = Y_{j^*}.$$

(b) Since the space $X_{j_{\min}}$ is compact by assumption and the space

$$X_{j_{\min}} \times \prod_{j_{\min} < j} X_j$$

cannot be covered by finitely many elements of $\mathcal{A}$, by the Tube Lemma there exists $p_{j_{\min}} \in X_{j_{\min}}$ such that

$$p_{j_{\min}} \times \prod_{j_{\min} < j} X_j$$

cannot be covered by finitely many elements of $\mathcal{A}$. Suppose $k \in J$, $k \in j_{\min}$, and we have chosen $p_i \in X_i$ for each $j < k$ so that the corresponding slices Y_j, with $j < k$, of part (a) cannot be covered by finitely many elements of $\mathcal{A}$. By part (a), the corresponding space

$$Z_k = (p_i)_{i<k} \times X_k \times \prod_{k<i} X_i$$

cannot be covered by finitely many elements of $\mathcal{A}$. Since the space X_k is compact, by the Tube Lemma there exists $p_k \in X_k$ such that

$$Z_k = (p_i)_{i<k} \times p_k \times \prod_{k<i} X_i = (p_i)_{i \leq k} \times \prod_{k<i} X_i$$

cannot be covered by finitely many elements of $\mathcal{A}$, as needed.

Remark 1: By part (b), if $\mathcal{A}$ is an open cover of X by basis elements, then $\mathcal{A}$ has a finite subcover. This implies that X is compact

Remark 2: The argument in part (b) above, as suggested by the book, is actually a little problematic. Here is a fix. Let $\mathcal{S}$ be the set of all subsets I of J such that if $j \in I$, $i \in J$, and $i < j$, then $i \in I$. Let $\mathcal{D}$ be the collection of all elements $(p_i)_{i \in I}$ of

$$\bigcup_{I \in \mathcal{S}} \prod_{j \in I} X_j$$

such that the corresponding slices Y_j, with $j \in I$, defined in part (a) cannot be covered by finitely many elements of $\mathcal{A}$. We define a partial ordering on $\mathcal{D}$ by

$$(p_i)_{i \in I} \prec (p'_i)_{i \in I'} \qquad \text{if} \qquad I \subsetneq I' \quad \text{and} \quad p_i = p'_i \ \forall\, i \in I.$$

By Zorn's Lemma (or the Maximum Principle), there exists a maximal simply ordered subset $\mathcal{D}^*$ of $\mathcal{D}$. Let I^* be the union of all sets I such that $(p_i)_{i \in I}$ is an element of $\mathcal{D}^*$. Since $\mathcal{D}^*$ is maximal, $(p_i)_{i \in I^*}$ is an element of $\mathcal{D}^*$ and is the largest element of $\mathcal{D}^*$. We claim that $I^* = J$. If not, take k to be the smallest element of $J - I^*$ and proceed as above to choose $p_k \in X_k$. The element $(p_i)_{i \in I^* \cup \{k\}}$ of $\mathcal{D}$ is larger that any element of $\mathcal{D}^*$, contrary to the assumption that $\mathcal{D}^*$ is a maximal simply ordered subcollection of $\mathcal{D}$.