MAT 530: Topology&Geometry, I
Fall 2005

Problem Set 2
Solution to Problem p127, #8

Let X be the subset of $\mathbb{R}^\omega$ consisting of all sequences

$$\mathbf{x} \equiv (x_1, x_2, \ldots)$$

such that $\sum x_i^2$ converges. Then the formula

$$d(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2\right]^{1/2}$$

defines a metric on X. On X we have the three topologies it inherits from the box, uniform, and product topologies on $\mathbb{R}^\omega$. We also have the topology given by the metric d, which we call the ℓ^2-topology.
(a) *Show that on X, we have the inclusions*

$$\text{box topology} \supset \ell^2\text{-topology} \supset \text{uniform topology}.$$

(b) *The set $\mathbb{R}^\infty$ of all sequences that are eventually zero is contained in X. Show that the four topologies that $\mathbb{R}^\infty$ inherits as subspace of X are all distinct.*
(c) *Compare the four topologies the Hilbert cube,*

$$H \equiv \prod_{n=1}^{\infty} [0, 1/n],$$

inherits as a subspace of X.

(a) Recall that the uniform metric is given by

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup\left\{\bar{d}(x_i, y_i) : i \in \mathbb{Z}^+\right\}, \qquad \text{where} \qquad \bar{d}(x_i, y_i) = \min\big(1, |x_i - y_i|\big).$$

Since $|x_i - y_i| \le d(\mathbf{x}, \mathbf{y})$ for all $i \in \mathbb{Z}^+$, it follows that

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) \le d(\mathbf{x}, \mathbf{y}) \ \ \forall\, \mathbf{x}, \mathbf{y} \in X \quad \Longrightarrow \quad B_d(\mathbf{x}, \delta) \subset B_{\bar{\rho}}(\mathbf{x}, \delta) \ \ \forall\, \mathbf{x} \in X,\ \delta \in \mathbb{R} \quad \Longrightarrow \quad \mathcal{T}_{\bar{\rho}} \subset \mathcal{T}_d.$$

This proves the second inclusion.

On the other hand, if $\mathbf{x} \in X$ and $\delta \in \mathbb{R}^+$,

$$\mathbf{x} \in \prod_{i=1}^{\infty} \big(x_i - 2^{-i}\delta, x_i + 2^{-i}\delta\big) \subset B_d(\mathbf{x}, \delta).$$

Since the set on the right is a general basis element for the ℓ^2-topology and the set in middle is open in the box topology, it follows that the box topology is finer than the ℓ^2-topology.

Remark: A priori, we need to show that given *any* element $\mathbf{y}\in B_d(\mathbf{x},\delta)$, there exists a subset $\mathcal{U}$ open in the box topology such that

$$\mathbf{y}\in\mathcal{U}\subset B_d(\mathbf{x},\delta).$$

Similarly, given *any* $\mathbf{y}\in B_{\bar\rho}(\mathbf{x},\delta)$, we need to show that there exists $\epsilon>0$ such that

$$\mathbf{y}\in B_d(\mathbf{y},\epsilon)\subset B_{\bar\rho}(\mathbf{x},\delta).$$

However, given any metric space (X,d), $x\in X$, $\delta\in\mathbb{R}^+$, and $y\in B_d(x,\delta)$, there exists $\epsilon\in\mathbb{R}^+$ such that

$$y\in B_d(y,\epsilon)\subset B_d(x,\delta).$$

Why? Thus, finding open sets around the centers of open balls as we have done above suffices. *Why?*

(b) By part (a) and Theorem 20.4,

$$\text{product topology}\subset\text{uniform topology}\subset\ell^2\text{-topology}\subset\text{box topology}$$

on X and thus on $\mathbb{R}^\infty\subset X$. Thus, in order to show that

$$\text{product topology}\subsetneq\text{uniform topology}\subsetneq\ell^2\text{-topology}\subsetneq\text{box topology},$$

it suffices to find a subset $\mathcal{U}$ of $\mathbb{R}^\infty$ containing $\mathbf{0}$ such that $\mathcal{U}$ is open in the uniform/ℓ^2-/box topology, but no basis element for the product/uniform/ℓ^2-topology containing $\mathbf{0}$ is contained in $\mathcal{U}$.

The ball $B_{\bar\rho}(\mathbf{0},1/2)$ contains no point $\mathbf{x}$ of X such that $x_i=1$ for some $i\in\mathbb{Z}^+$. On the other hand, if

$$V\equiv\prod_{i=1}^{\infty}V_i$$

is a basis element for the product topology on $\mathbb{R}^\omega$, then $V_i=\mathbb{R}$ for some $i\in\mathbb{Z}^+$ (in fact, for all but finitely many i's). Then,

$$(0,\ldots,0,x_i=1,0,0,\ldots)\in V\cap\mathbb{R}^\infty\qquad\Longrightarrow\qquad V\cap\mathbb{R}^\infty\not\subset B_{\bar\rho}(\mathbf{0},1/2)\supset B_{\bar\rho}(\mathbf{0},1/2)\cap\mathbb{R}^\infty.$$

This shows that the first inclusion above cannot be an equality.

We next show that for all $\delta>0$

$$B_{\bar\rho}(\mathbf{0},\delta)\cap\mathbb{R}^\infty\not\subset B_d(\mathbf{0},1/2).$$

Choose $N\in\mathbb{Z}^+$ such that $N>1/\delta^2$. Let

$$\mathbf{x}=\big(\delta/2,\delta/2,\ldots,x_N=\delta/2,0,0,\ldots\big)\in B_{\bar\rho}(\mathbf{0},\delta)\cap\mathbb{R}^\infty.$$

Then,

$$d(\mathbf{0},\mathbf{x})=\big(N(\delta/2)^2\big)^{1/2}=N^{1/2}(\delta/2)>1/2\quad\Longrightarrow\quad\mathbf{x}\notin B_d(\mathbf{0},1/2).$$

Finally, we show that for all $\delta > 0$

$$B_d(\mathbf{0},\delta)\cap\mathbb{R}^\infty \not\subset \mathcal{U} \equiv \prod_{i=1}^{\infty}(-1/i, 1/i).$$

Choose $N > 2/\delta$. Then,

$$\mathbf{x} = \big(0,\ldots,0, x_N = \delta/2, 0, 0, \ldots\big) \in B_d(\mathbf{0},\delta)\cap\mathbb{R}^\infty.$$

However, $\mathbf{x} \notin \mathcal{U}$, since $x_N > 1/N$.

(c) We will show that on H

$$\text{product topology} = \text{uniform topology} = \ell^2\text{-topology} \subsetneq \text{box topology}.$$

By (a) and Theorem 20.4, it is sufficient to show that the product topology contains the ℓ^2-topology and is different from the box topology.

Suppose $\mathbf{x} \in H$ and $\delta \in \mathbb{R}^+$. Choose $N \in \mathbb{Z}^+$ and then $\epsilon \in \mathbb{R}^+$ so that

$$\sum_{i=N+1}^{\infty} 1/i^2 < (\delta/2)^2 \qquad \text{and} \qquad N\epsilon < \delta/2.$$

The set

$$V \equiv \left(\prod_{i=1}^{i=N}(x_i-\epsilon, x_i+\epsilon) \times \prod_{i=N+1}^{\infty}\mathbb{R}\right) \cap H = \prod_{i=1}^{i=N}\big((x_i-\epsilon, x_i+\epsilon)\cap[0,1/i]\big) \times \prod_{i=N+1}^{\infty}[0,1/i]$$

contains $\mathbf{x}$ and is open in H in the product topology. If $\mathbf{y} \in V$, then

$$d(\mathbf{x},\mathbf{y}) \equiv \left[\sum_{i=1}^{\infty}(x_i-y_i)^2\right]^{1/2} \le \left[N\epsilon^2 + \sum_{i=N+1}^{\infty}1/i^2\right]^{1/2} \le N\epsilon + \delta/2 < \delta$$

$$\Longrightarrow \qquad \mathbf{y} \in B_d(\mathbf{x},\delta) \qquad \Longrightarrow \qquad \mathbf{x} \in V \subset B_d(\mathbf{x},\delta).$$

We have now verified the first claim.

If V is a nonempty basis element for the product topology on H,

$$V = \prod_{i=1}^{i=N} V_i \times \prod_{i=N+1}^{\infty}[0,1/i]$$

for some $N \in \mathbb{Z}^+$ and V_i nonempty and open in $[0,1/i]$. In particular, V must contain a point $\mathbf{x}$ such that $x_i = 1/i$ for some i. On the other hand, the set

$$\mathcal{U} = \prod_{i=1}^{\infty}[0, 2^{-i}]$$

is open in the box topology on H and contains no such point. Thus, $\mathcal{U}$ contains no subset which is nonempty and open in the product topology. Since $\mathcal{U}$ is nonempty, we conclude that the box topology is different from the product topology on H.