MAT 530: Topology&Geometry, I
Fall 2005

Final Exam Solutions

Part I (*choose 2 problems from 1,2, and 3*)

1. *Suppose X is a $T1$ topological space. Show directly from the definitions that X is normal if and only if for every closed subset A of X and open subset $\mathcal{U}$ of X such that $A\subset\mathcal{U}$ there exists an open subset V of X such that $A\subset V$ and $\bar{V}\subset\mathcal{U}$.*

Suppose X is normal, A is a closed subset of X, and $\mathcal{U}$ is an open subset of X such that $A\subset\mathcal{U}$. Since $\mathcal{U}$ is open in X, the set $B\equiv X-\mathcal{U}$ is closed in X. Since $A\subset\mathcal{U}$,

$$A\cap B = A\cap(X-\mathcal{U}) = A-\mathcal{U} = \emptyset.$$

Thus, A and B are disjoint closed subsets in X. Since X is normal, there exist disjoint open subsets V and W in X such that $A\subset V$ and $B\subset W$. Since $V\cap W=\emptyset$ and W is an open set,

$$\bar{V}\cap W=\emptyset \qquad\Longrightarrow\qquad \bar{V}-\mathcal{U}=\bar{V}\cap(X-\mathcal{U})=\bar{V}\cap B\subset\bar{V}\cap W=\emptyset \qquad\Longrightarrow\qquad \bar{V}\subset\mathcal{U},$$

as needed.

Suppose X is a $T1$-topological space and for every closed subset A of X and open subset $\mathcal{U}$ of X such that $A\subset\mathcal{U}$ there exists an open subset V of X such that $A\subset V$ and $\bar{V}\subset\mathcal{U}$. Let A and B be disjoint closed subsets of X. Since B is closed, the set $\mathcal{U}\equiv X-B$ is open in X. Furthermore,

$$A-\mathcal{U}=A\cap(X-\mathcal{U})=A\cap B=\emptyset \qquad\Longrightarrow\qquad A\subset\mathcal{U}.$$

Thus, there exists an open subset V of X such that $A\subset V$ and $\bar{V}\subset\mathcal{U}$. Let $W=X-\bar{V}$. Since $\bar{V}$ is closed in X, W is open in X. Furthermore,

$$V\cap W=V\cap(X-\bar{V})=V-\bar{V}=\emptyset \qquad\text{and}\qquad B=X-\mathcal{U}\subset X-\bar{V}=W.$$

Thus, V and W are disjoint open subsets of X containing A and B, respectively. We conclude that X is normal.

2. *Let X be the subspace of $\mathbb{R}^2$ given by*

$$X=(I\times 0)\cup(0\times I)\cup\bigcup_{n\in\mathbb{Z}^+}(1/n)\times I, \qquad \textit{where} \qquad I=[0,1].$$

Show that the retraction $r: X\longrightarrow 0\times 1$ is a homotopy equivalence, but 0×1 is not a deformation retract of X (i.e. there exists no homotopy $H: X\times I\longrightarrow X$ between id_X and r such that $H(0\times 1,t)=0\times 1$ for all t).

The space X is an infinite comb:

We denote by p the point 0×1 and by $i\colon \{p\}\longrightarrow X$ the inclusion map. We need to show that the maps

$$r\circ i\colon \{p\}\longrightarrow \{p\} \qquad \text{and} \qquad i\circ r\colon X\longrightarrow X$$

are homotopic to $\mathrm{id}_{\{p\}}$ and id_X, respectively. In fact, the first map is $\mathrm{id}_{\{p\}}$. To homotop id_X to r, first retract the vertical line segments *at the same rate* down to $I\times 0$, then retract $I\times 0$ to 0×0, and finally move 0×0 to p. Formally,

$$H\colon X\times I\longrightarrow X, \qquad H\big(u\times v,t\big)=\begin{cases}(u,(1-3t)v), & \text{if } t\in[0,1/3];\\ ((2-3t)u,0), & \text{if } t\in[1/3,2/3];\\ (0,3t-2), & \text{if } t\in[2/3,1].\end{cases}$$

During this homotopy the point p gets moved around.

It remains to show that if $H\colon X\times I\longrightarrow X$ is any homotopy from id_X to r, then $H(p,t^*)\neq p$ for some $t^*\in[0,1]$. We show in the next paragraph that for each $n\in\mathbb{Z}^+$ there exists $t_n\in[0,1]$ such that

$$H\big((1/n)\times 1,t_n\big)=(1/n)\times 0,$$

i.e. the top of the vertical line segment $(1/n)\times I$ must pass through the bottom point of the line segment in order to get to p. Since I is a compact set, a subsequence of $(t_n)_{n\in\mathbb{Z}^+}$ converges to some $t^*\in I$. Since the sequence $(1/n)\times 1$ converges to p in X, a subsequence of $((1/n)\times 1,t_n)$ converges to (p,t^*). Since H is a continuous function, it follows that $H(p,t^*)$ is the limit of a subsequence $H((1/n)\times 1,t_n)$. Since $H\big((1/n)\times 1,t_n\big)=(1/n)\times 0$ for all $n\in\mathbb{Z}^+$, we conclude that $H(p,t^*)=(0,0)\neq p$, as claimed.

Finally, we show that for each $n\in\mathbb{Z}^+$ there exists $t_n\in[0,1]$ such that $H\big((1/n)\times 1,t_n\big)=(1/n)\times 0$. Suppose not. Then, the image of the map

$$f\colon I\longrightarrow X, \qquad f(t)=H\big((1/n)\times 1,t\big),$$

is contained in the set

$$X-\{(1/n)\times 0\}=(1/n)\times(0,1]\cup\big(X-(1/n)\times[0,1]\big).$$

The sets $(1/n)\times(0,1]$ and $X-(1/n)\times[0,1]$ are open and disjoint. Since I is connected, so is $f(I)$ and thus $f(I)$ is contained either in $(1/n)\times(0,1]$ or in $X-(1/n)\times[0,1]$. However,

$$f(0)=H\big((1/n)\times 1,0\big)=\mathrm{id}_X\big((1/n)\times 1\big)=(1/n)\times 1\in(1/n)\times(0,1] \qquad \text{and}$$
$$f(1)=H\big((1/n)\times 1,1\big)=r\big((1/n)\times 1\big)=p\in X-(1/n)\times[0,1].$$

Thus, $f(I)\not\subset X-\{(1/n)\times 0\}$.

3. (a) *If p, q, and r are points in $\mathbb{R}^2$, let*

$$\mathcal{P}(p,q) = \big\{f\in\mathcal{C}(I,\mathbb{R}^2)\colon f(0)=p, f(1)=q\big\} \qquad \textit{and} \qquad \mathcal{P}(p,q;r) = \big\{f\in\mathcal{P}(p,q)\colon f^{-1}(r)=\emptyset\big\}.$$

Show that if $r\neq p,q$, then $\mathcal{P}(p,q;r)$ is open and dense in $\mathcal{P}(p,q)$ in the uniform topology ($I=[0,1]$ and $\mathcal{C}(I,\mathbb{R}^2)$ is the space of continuous functions from I to $\mathbb{R}^2$).
(b) *If $Y\subset\mathbb{R}^2$ is countable, show that $\mathbb{R}^2-Y$ is path-connected.*

The space $\mathcal{P}(p,q)$ consists of paths from p to q in $\mathbb{R}^2$; $\mathcal{P}(p,q;r)$ consists of the paths from p to q that do not pass through the point r. The uniform topology on $\mathcal{P}(p,q)$ is the topology corresponding to the uniform metric on $\mathcal{P}(p,q)$:

$$\rho(f,g) = \sup\big\{\big|f(s)-g(s)\big|\colon s\in I\big\},$$

where $|x-y|$ denotes the standard (round) norm on $\mathbb{R}^2$. A basis element in this topology is

$$B_\rho(f,\epsilon) \equiv \big\{g\in\mathcal{P}(p,q)\colon \rho(f,g)<\epsilon\big\} = \big\{g\in\mathcal{P}(p,q)\colon \big|f(s)-g(s)\big|<\epsilon \ \forall\, s\in I\big\}.$$

The second equality holds because I is compact and thus

$$\sup\big\{\big|f(s)-g(s)\big|\colon s\in I\big\} = \big|f(s^*)-g(s^*)\big| \qquad \text{for some} \qquad s^*\in I.$$

(a) First we show that $\mathcal{P}(p,q;r)$ is open in $\mathcal{P}(p,q)$, i.e. for every $f\in\mathcal{P}(p,q;r)$ there exists $\epsilon>0$ such that $B_\rho(f,\epsilon)\subset\mathcal{P}(p,q;r)$. Since f is continuous and I is compact, $f(I)$ is a compact subset of $\mathbb{R}^2$. Since the compact set $f(I)$ is contained in the open set $\mathbb{R}^2-\{r\}$, there exists $\epsilon>0$ such that

$$\bigcup_{s\in I}\big\{y\in\mathbb{R}^2\colon |y-f(s)|<\epsilon\big\} \subset \mathbb{R}^2-\{r\}.$$

Then, for all $g\in B_\rho(f,\epsilon)$

$$|f(s)-g(s)|<\epsilon \ \ \forall\, s\in I \quad\Longrightarrow\quad g(s)\in\mathbb{R}^2-\{r\} \ \ \forall\, s\in I \quad\Longrightarrow\quad g^{-1}(r)=\emptyset \quad\Longrightarrow\quad g\in\mathcal{P}(p,q;r).$$

Thus, $B_\rho(f,\epsilon)\subset\mathcal{P}(p,q;r)$.

We next show that $\mathcal{P}(p,q;r)$ is dense in $\mathcal{P}(p,q)$, i.e. for every $f\in\mathcal{P}(p,q)$ and $\epsilon>0$

$$B_\rho(f,\epsilon)\cap\mathcal{P}(p,q;r)\neq\emptyset.$$

Let $\delta=|r-q|>0$. We can assume that $\epsilon<\delta$. Since f is continuous and I is compact, f is uniformly continuous. In particular, there exist

$$\begin{aligned} &t_0,t_1,\ldots,t_n\in I \qquad \text{s.t.} \qquad t_0=0<t_1<\ldots<t_{n-1}<t_n=1 \\ &\text{and} \qquad \big|f(s)-f(t)\big|<\epsilon/3 \quad \forall\ s,t\in[t_{k-1},t_k],\ k=1,2,\ldots,n. \end{aligned}$$

Let $g_0=f|_{t_0}$. Suppose $0\le k\le n-1$ and we have constructed a continuous function

$$\begin{gathered} g_k\colon[0,t_k]\longrightarrow\mathbb{R}^2 \qquad \text{s.t.} \qquad g_k^{-1}(r)=\emptyset, \quad g_k(0)=p, \\ \big|g_k(t_k)-f(t_k)\big|<\epsilon/3, \quad \big|g_k(t_k)-f(t_{k+1})\big|<\epsilon/3, \quad \big|g_k(t)-f(t)\big|<\epsilon/3 \ \ \forall\, t\in[0,t_k]. \end{gathered}$$

This is the case for $k=0$, since $g_0(0)=f(0)\neq r$. Let ℓ_k be the line through the points r and $g(t_k)$. If $k+1\neq n$ (i.e. $t_{k+1}\neq 1$), choose

$$y_{k+1}\in\mathbb{R}^2-\ell_k \quad \text{s.t.} \quad |y_{k+1}-g_k(t_k)|<\epsilon/3, \quad |y_{k+1}-f(t_{k+1})|<\epsilon/3, \quad |y_{k+1}-f(t_{k+2})|<\epsilon/3.$$

This is possible, since $|f(t_{k+1})-f(t_{k+2})|<\epsilon/3$. If $k+1=n$, take $y_{k+1}=f(1)=q$. Define

$$g_{k+1}\colon[0,t_{k+1}]\longrightarrow\mathbb{R}^2 \qquad \text{by} \qquad g_{k+1}(t)=\begin{cases} g_k(t), & \text{if } t\in[0,t_k];\\ \frac{t_{k+1}-t}{t_{k+1}-t_k}g_k(t_k)+\frac{t-t_k}{t_{k+1}-t_k}y_{k+1}, & \text{if } t\in[t_k,t_{k+1}].\end{cases}$$

If $k+1\neq n$, then $y_{k+1}\notin\ell_k$. Thus, $g_{k+1}(t)\notin\ell_k$ for all $t\in(t_k,t_{k+1}]$ and $g_{k+1}(t)\neq r$ for all $t\in(t_k,t_{k+1}]$. If $k+1=n$, then

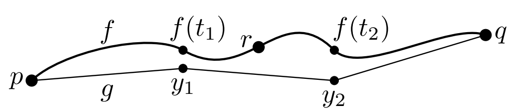
$$\begin{aligned} y_{k+1}=f(1), \quad |f(t_k)-f(1)|<\epsilon<\delta \qquad &\Longrightarrow \qquad \big|g_{k+1}(t)-f(1)\big|<\delta \quad \forall\, t\in[t_k,t_{k+1}]\\ &\Longrightarrow \qquad g_{k+1}(t)\neq r \quad \forall\, t\in[t_k,t_{k+1}]. \end{aligned}$$

Thus, $g_{k+1}^{-1}(r)=\emptyset$. Since $g_{k+1}(t_{k+1})=y_{k+1}$, the two inductive requirements on $g_{k+1}(t_{k+1})$ are satisfied. Finally, for all $t\in[t_k,t_{k+1}]$,

$$\big|g_{k+1}(t)-f(t)\big|\le\big|g_{k+1}(t)-g_k(t_k)\big|+\big|g_k(t_k)-f(t_k)\big|+\big|f(t_k)-f(t)\big|<3(\epsilon/3)=\epsilon.$$

After n steps we obtain a piecewise linear path $g\equiv g_n$ from p to q such that $q^{-1}(r)=\emptyset$ and $|f(t)-g(t)|<\epsilon$ for all $t\in I$, i.e.

$$g\in B_\rho(f,\epsilon)\cap\mathcal{P}(p,q;r).$$



(b) Let $p,q\in\mathbb{R}^2-Y$ be any two points. The evaluation map

$$\mathrm{ev}_{0,1}\colon\mathcal{C}(I,\mathbb{R}^2)\longrightarrow\mathbb{R}^2\times\mathbb{R}^2, \qquad \mathrm{ev}_{0,1}(f)=\big(f(0),f(1)\big),$$

is continuous in the uniform topology on $\mathcal{C}(I,\mathbb{R}^2)$. Thus,

$$\mathcal{P}(p,q)=\mathrm{ev}_{0,1}^{-1}(p\times q)\subset\mathcal{C}(I,\mathbb{R}^2)$$

is a closed subset. Since $\big(\mathcal{C}(I,\mathbb{R}^2),\rho\big)$ is a complete metric space, so is $\big(\mathcal{P}(p,q),\rho\big)$. By the Baire category theorem, $\mathcal{P}(p,q)$ is a Baire space. The space of paths in $\mathbb{R}^2-Y$ from p to q is

$$\mathcal{P}(p,q;Y)=\bigcap_{r\in Y}\mathcal{P}(p,q;r).$$

By part (a), $\mathcal{P}(p,q;r)$ is a dense open subset of $\mathcal{P}(p,q)$ for all $r\in Y\subset\mathbb{R}^2-\{p,q\}$. Since $\mathcal{P}(p,q)$ is a Baire space and Y is countable, it follows that $\mathcal{P}(p,q;Y)$ is dense in $\mathcal{P}(p,q)$. In particular, $\mathcal{P}(p,q;Y)$ is not empty, i.e. there exists a path in $\mathbb{R}^2-Y$ from p to q. We conclude that $\mathbb{R}^2-Y$ is path-connected.

Part II (*choose 2 problems from 4,5, and 6*)

4. *Determine the number of coverings of the space* $\mathbb{R}P^2 \times \mathbb{R}P^2$ *(up to equivalence, with path-connected covering space).*

Since $\pi_1(\mathbb{R}P^2)=\mathbb{Z}_2$,

$$\pi_1\big(\mathbb{R}P^2 \times \mathbb{R}P^2\big) = \pi_1(\mathbb{R}P^2) \times \pi_1(\mathbb{R}P^2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Since $\mathbb{R}P^2 \times \mathbb{R}P^2$ has a universal cover, i.e. $S^2 \times S^2$, the equivalence classes of coverings of $\mathbb{R}P^2 \times \mathbb{R}P^2$ correspond to the conjugacy classes of subgroups of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since this group is abelian, the number of coverings of $\mathbb{R}P^2 \times \mathbb{R}P^2$ is the number of subgroups of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. There are 5 such subgroups: the trivial subgroup, the entire group, and the three subgroups consisting of the zero element and one of the nonzero elements. So, the answer is 5.

Remark: The trivial subgroup and the entire group correspond to the universal cover $S^2 \times S^2$ and the space $\mathbb{R}P^2 \times \mathbb{R}P^2$ itself, respectively. The two proper subgroups $\pi_1\big(\mathbb{R}P^2 \times p_2\big)$ and $\pi_1\big(p_1 \times \mathbb{R}P^2\big)$ of $\pi_1\big(\mathbb{R}P^2 \times \mathbb{R}P^2\big)$ correspond to $\mathbb{R}P^2 \times S^2$ and $S^2 \times \mathbb{R}P^2$. Finally, the "diagonal" two-element subgroup of $\pi_1\big(\mathbb{R}P^2 \times \mathbb{R}P^2\big)$ corresponds to

$$\big(S^2 \times S^2\big)/\{\pm 1\} = \mathbb{R}\widetilde{P^2 \times \mathbb{R}}P^2, \qquad \text{where} \qquad (-1) \cdot (x_1 \times x_2) = (-x_1) \times (-x_2);$$

$\mathbb{R}\widetilde{P^2 \times \mathbb{R}}P^2$ is the orientable double-cover of $\mathbb{R}P^2 \times \mathbb{R}P^2$. The fiber of this covering map over a point of $\mathbb{R}P^2 \times \mathbb{R}P^2$ consists of the two orientations at that point. In general, given a connected manifold M the double cover $\tilde{M}$ of M defined in this way is connected if and only if M is *not* orientable.

5. *Show that* $\mathbb{R}P^3$ *is not homeomorphic to a product of two manifolds of positive dimensions.*

Suppose $\mathbb{R}P^3$ is homeomorphic to $X \times Y$, where X and Y are manifolds of positive dimensions. Since the dimension of $\mathbb{R}P^3$ is 3, the dimension of X must be 1 and the dimension of Y must be 2 (or vice versa). Since $\mathbb{R}P^3$ is compact and connected, so is X. Thus, X is a compact connected 1-dimensional manifold, i.e. $X = S^1$. Thus,

$$\pi_1(\mathbb{R}P^3) = \pi_1(X) \times \pi_1(Y) = \mathbb{Z} \times \pi_1(Y).$$

However, this is impossible, since $\pi_1(\mathbb{R}P^3) = \mathbb{Z}_2$ is finite.

6. *Show that if* $f\colon S^1 \longrightarrow S^1$ *is null-homotopic, then* $f(x) = -x$ *for some* $x \in S^1$.

Let $g\colon S^1 \longrightarrow S^1$ be the map given by $g(x) = -f(x)$. Since f is null-homotopic, so is g. Thus, g extends to a continuous map

$$h\colon B^2 \longrightarrow S^1 \subset B^2.$$

By Brouwer fixed-point theorem, h has a fixed point, i.e. $h(y) = y$ for some $y \in B^2$. Since $h(x) \in S^1$ for all $x \in B^2$, $y \in S^1$. Thus,

$$y = h(y) = g(y) = -f(y) \qquad \Longrightarrow \qquad f(y) = -y.$$

Part III (*choose 1 problem from 7 and 8*)

7. *Let T denote the standard torus. Show that two continuous maps $f, g\colon (T, x_0) \longrightarrow (T, y_0)$ are homotopic if and only if the homomorphisms*

$$f_*, g_*\colon \pi_1(T, x_0) \longrightarrow \pi_1(T, y_0)$$

are equal. (In this case, homotopic means homotopic as maps of pointed spaces, so the homotopy must always map x_0 to y_0).

If $H : T \times I \longrightarrow T$ is a homotopy from f to g such that $H(x_0, t) = y_0$ for all $t \in I$, then the homomorphisms

$$f_*, g_*\colon \pi_1(T, x_0) \longrightarrow \pi_1(T, y_0)$$

are equal. In fact, if $\alpha\colon I \longrightarrow T$ is a loop based at x_0, then $H\circ\{\alpha\times\mathrm{id}\}$ is a path-homotopy from $f\circ\alpha$ to $g\circ\alpha$.

Suppose the homomorphisms $f_*, g_* : \pi_1(T, x_0) \longrightarrow \pi_1(T, y_0)$ are equal. Since $T = S^1 \times S^1$, we can write $x_0 = x_{0,1} \times x_{0,2}$. Let

$$p\colon \big(\mathbb{R}\times\mathbb{R}, 0\times 0\big) \longrightarrow \big(S^1\times S^1, x_0\big) \qquad p(u,v) = (x_{0,1}e^{2\pi i u}) \times (x_{0,2}e^{2\pi i v}),$$

be the universal covering. Let $q = p|_{I\times I}$. Choose a point $\tilde{y}_0 \in p^{-1}(y_0)$. Let

$$\tilde{q}_f, \tilde{q}_g\colon \big(I\times I, 0\times 0\big) \longrightarrow \big(\mathbb{R}\times\mathbb{R}, \tilde{y}_0\big)$$

be the lifts of the maps

$$f\circ q, g\circ q : \big(I\times I, 0\times 0\big) \longrightarrow (T, y_0),$$

respectively, over p:

$$\begin{tikzcd}
(I^2, 0) \arrow[r, dashed, "\tilde{q}_f"] \arrow[d, "q"] & (\mathbb{R}^2, \tilde{y}_0) \arrow[d, "p"] \\
(T, x_0) \arrow[r, "f"] & (T, y_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
(I^2, 0) \arrow[r, dashed, "\tilde{q}_g"] \arrow[d, "q"] & (\mathbb{R}^2, \tilde{y}_0) \arrow[d, "p"] \\
(T, x_0) \arrow[r, "g"] & (T, y_0)
\end{tikzcd}$$

Let

$$\tilde{H}\colon \big(I\times I, 0\times 0\big)\times I \longrightarrow \big(\mathbb{R}\times\mathbb{R}, \tilde{y}_0\big), \qquad H(z,t) = (1-t)\,\tilde{q}_f(z) + t\,\tilde{q}_g(z),$$

be the straight-line homotopy from $\tilde{q}_f$ to $\tilde{q}_g$ in $\mathbb{R}\times\mathbb{R}$. Then,

$$p\circ\tilde{H}\colon \big(I\times I, 0\times 0\big)\times I \longrightarrow \big(T, y_0\big)$$

is a homotopy from $f\circ q$ to $g\circ q$. In the next two paragraphs we will use the assumption that $f_* = g_*$ to show that

$$\begin{aligned}
&\tilde{H}\big(1\times v, t\big) - \tilde{H}\big(0\times v, t\big) \in \mathbb{Z}^2 \subset \mathbb{R}^2, \quad \tilde{H}\big(u\times 1, t\big) - \tilde{H}\big(u\times 0, t\big) \in \mathbb{Z}^2 \subset \mathbb{R}^2 \qquad \forall\, u, v, t \in I \\
\Longrightarrow \quad &\{p\circ\tilde{H}\}\big(0\times v, t\big) = \{p\circ\tilde{H}\}\big(1\times v, t\big), \quad \{p\circ\tilde{H}\}\big(u\times 0, t\big) = \{p\circ\tilde{H}\}\big(u\times 1, t\big) \qquad \forall\, u, v, t \in I.
\end{aligned}$$

Thus, $\{p\circ\tilde{H}\}$ descends to a map on the quotient

$$H\colon X=\big((I\times I)\times I\big)/\!\sim\;\longrightarrow T,\quad\text{where}\quad \big(0\times v,t\big)\sim(1\times v,t\big),\quad \big(u\times 0,t\big)\sim\big(u\times 1,t\big)\quad \forall\, u,v,t\in I.$$

This map is continuous in the quotient topology. With this topology, X is homeomorphic to $T\times I$. The quotient projection map is

$$q\times\mathrm{id}:(I\times I)\times I\longrightarrow T\times I.$$

(we have simply identified the two vertical edges in the square $I\times I$ and the two horizontal edges in $I\times I$). Thus, we have found a continuous map $H\colon T\times I\longrightarrow T$ such that

$$\begin{aligned}
&p\circ\tilde{H}=H\circ(q\times\mathrm{id}\,),\quad \tilde{H}|_{I^2\times 0}=\tilde{q}_f,\quad \tilde{H}|_{I^2\times 1}=\tilde{q}_g,\quad \tilde{H}(x_0,t)=\tilde{y}_0\ \ \forall\, t\in I \qquad\Longrightarrow\\
&H\circ\{q\times\mathrm{id}\,\}|_{I^2\times 0}=p\circ\tilde{H}|_{I^2\times 0}=p\circ\tilde{q}_f=f\circ q,\quad H\circ\{q\times\mathrm{id}\,\}|_{I^2\times 1}=p\circ\tilde{H}|_{I^2\times 1}=p\circ\tilde{q}_g=g\circ q\\
&\{p\circ\tilde{H}\}|_{(0\times 0)\times I}=y_0 \qquad\Longrightarrow\qquad H|_{T\times 0}=f,\quad H|_{T\times 1}=g,\quad H|_{x_0\times 0}=y_0.
\end{aligned}$$

$$\begin{tikzcd}
(I^2,0)\times I \arrow[r, "\tilde{H}"] \arrow[d, "q\times\mathrm{id}"] & (\mathbb{R}^2,\tilde{y}_0) \arrow[d, "p"] \\
(T,x_0)\times I \arrow[r, dashed, "H"] & (T,y_0)
\end{tikzcd}$$

We conclude that H is a homotopy from f and g.

It remains to show that

$$\tilde{H}\big(1\times v,t\big)-\tilde{H}\big(0\times v,t\big)\in\mathbb{Z}^2\subset\mathbb{R}^2,\quad \tilde{H}\big(u\times 1,t\big)-\tilde{H}\big(u\times 0,t\big)\in\mathbb{Z}^2\subset\mathbb{R}^2\qquad \forall\ u,v,t\in I.$$

We define loops α,β in T based at x_0 by

$$\alpha(s)=q(s,0)\qquad\text{and}\qquad \beta(s)=q(0,s).$$

By the uniqueness of the lifts,

$$\tilde{\alpha}_f(s)=\tilde{q}_f(s,0),\quad \tilde{\alpha}_g(s)=\tilde{q}_g(s,0),\quad \tilde{\beta}_f(s)=\tilde{q}_f(0,s),\quad\text{and}\quad \tilde{\beta}_g(s)=\tilde{q}_g(0,s)$$

are the lifts of the loops

$$f\circ\alpha,g\circ\alpha,f\circ\beta,g\circ\beta\colon(I,0)\longrightarrow(T^2,y_0),$$

respectively, beginning at $\tilde{y}_0$. Since $f_*=g_*$,

$$\begin{aligned}
&[f\circ\alpha]=[g\circ\alpha]\in\pi_1(T,y_0)\qquad\text{and}\qquad [f\circ\beta]=[g\circ\beta]\in\pi_1(T,y_0)\\
\Longrightarrow\qquad &\tilde{\alpha}_f(1)-\tilde{y}_0=\tilde{\alpha}_g(1)-\tilde{y}_0\qquad\text{and}\qquad \tilde{\beta}_f(1)-\tilde{y}_0=\tilde{\beta}_g(1)-\tilde{y}_0.
\end{aligned}$$

We denote the two vectors above by n_α and n_β, respectively. Since α_f and β_f are loops in T,

$$n_\alpha,n_\beta\in\mathbb{Z}^2\subset\mathbb{R}^2.$$

Since $q(0,v)=q(1,v)$ for all $v\in I$,

$$p\big(\tilde{q}_f(0,v)\big) = f\big(q(0,v)\big) = f\big(q(1,v)\big) = p\big(\tilde{q}_f(1,v)\big).$$

Thus, $\tilde{q}_f(1,v)-\tilde{q}_f(0,v)\in\mathbb{Z}^2$. Since the function

$$h\colon I \longrightarrow \mathbb{Z}^2 \subset \mathbb{R}^2, \qquad h(v) = \tilde{q}_f(1,v)-\tilde{q}_f(0,v),$$

is continuous, the set $\mathbb{Z}^2$ is discreet, and I is connected, it follows that h is a constant function. In particular,

$$\tilde{q}_f(1,v)-\tilde{q}_f(0,v) = \tilde{q}_f(1,0)-\tilde{q}_f(0,0) = \tilde{\alpha}_f(1)-\tilde{\alpha}_f(0) = \tilde{\alpha}_f(1)-\tilde{y}_0 = n_\alpha \qquad \forall\, v\in I.$$

By the same argument, we find that

$$\tilde{q}_g(1,v)-\tilde{q}_g(0,v) = \tilde{\alpha}_g(1)-\tilde{\alpha}_g(0) = n_\alpha \qquad \forall\, v\in I,$$
$$\tilde{q}_f(u,1)-\tilde{q}_f(u,0) = \tilde{\beta}_f(1)-\tilde{\beta}_f(0) = n_\beta \quad \text{and} \quad \tilde{q}_g(u,1)-\tilde{q}_g(u,0) = \tilde{\beta}_g(1)-\tilde{\beta}_g(0) = n_\beta \quad \forall\, u\in I.$$

Thus,

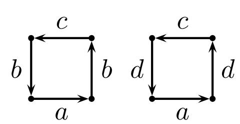
$$\begin{aligned}
\tilde{H}\big(1\times v,t\big) - \tilde{H}\big(0\times v,t\big) &= \big((1-t)\,\tilde{q}_f(1,v) + t\,\tilde{q}_g(1,v)\big) - \big((1-t)\,\tilde{q}_f(0,v) + t\,\tilde{q}_g(0,v)\big)\\
&= (1-t)\big(\tilde{q}_f(1,v)-\tilde{q}_f(0,v)\big) + t\big(\tilde{q}_g(1,v)-\tilde{q}_g(0,v)\big)\\
&= (1-t)n_\alpha + t\,n_\alpha = n_\alpha \in \mathbb{Z}^2.
\end{aligned}$$

Similarly,

$$\tilde{H}\big(u\times 1,t\big) - \tilde{H}\big(u\times 0,t\big) = n_\beta \in \mathbb{Z}^2,$$

as claimed.

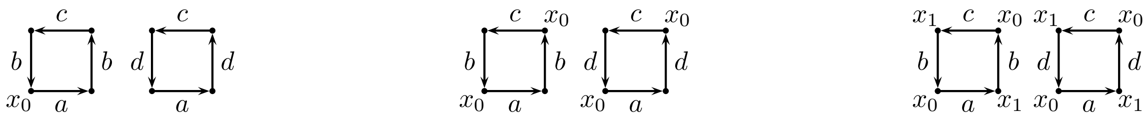
8. *Determine the homeomorphism type of $X=X_{abcb,adcd}$, i.e. the topological space represented by the diagram*



In particular, explain why X is a compact connected surface and determine to which of the standard surfaces X is homeomorphic to.

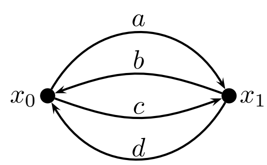
The space X is compact because it is a quotient of a compact space, the union of the two squares. It is connected because an edge of one of the squares is identified with an edge of the other square. Thus, X is a union of two connected subsets (images of the two squares under the quotient map) with a nonempty intersection. Since every label, a, b, c, and d, appears exactly twice, the labeling scheme is regular and thus X is a surface.

By the Classification Theorem for compact connected surfaces, the homeomorphism type of X is thus determined by $H_1(X)$. In order to compute $\pi_1(X)$ (and then $H_1(X)$), we first need to determine the image of the vertices under the quotient map, i.e. which of the vertices are identified. We begin by labeling the bottom left vertex of the first square x_0:



Since the bottom edges of the two squares are identified as indicated by the arrows, the bottom left vertex of the second square is also mapped to x_0. Since the two side edges in each of the squares are identified after a flip, the bottom left and upper right vertices in each of the squares are identified and thus should also be labeled x_0; see the middle diagram above. We can't get to any of the other four vertices in this way. So, they will be mapped to a different point or points in X. We label the bottom right vertex in the first square by x_1. Proceeding as above, we find that the remaining three vertices are identified with x_1 and should also be labeled x_1, as in the last diagram above.

By the last diagram above, the image A of the boundaries of the two squares consists of two points, x_0 and x_1, two line segments running from x_0 to x_1 (a and c), and two line segments running from x_1 to x_0 (b and d):



From this picture, we see that $\pi_1(A, x_0)$ is a free group on three generators:

$$\alpha = ab, \quad \beta = ac^{-1}, \quad \gamma = ad \qquad \text{i.e.} \qquad \pi_1(A, x_0) = \langle \alpha, \beta, \gamma | \cdot \rangle.$$

All other simple loops based at x_0 can be written as products of these three generators. The space X is obtained from A by adjoining two two-cells, the two squares. Thus, $\pi_1(X, x_0)$ is obtained from $\pi_1(A, x_0)$ by killing off the images of the boundaries of the squares, i.e.:

$$w_1 = abcb = \alpha\beta^{-1}\alpha \qquad \text{and} \qquad w_2 = adcd = \gamma\beta^{-1}\gamma.$$

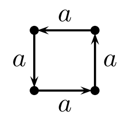
It follows that

$$\begin{aligned}
\pi_1(X, x_0) = \langle \alpha, \beta, \gamma | w_1, w_2 \rangle = \langle \alpha, \beta, \gamma | \alpha\beta^{-1}\alpha, \gamma\beta^{-1}\gamma \rangle \qquad \Longrightarrow \\
H_1(X) &= \big(\mathbb{Z}[\alpha] \oplus \mathbb{Z}[\beta] \oplus \mathbb{Z}[\gamma]\big) / \big(\mathbb{Z}[2\alpha - \beta] \oplus \mathbb{Z}[2\gamma - \beta]\big) \\
&= \big(\mathbb{Z}[\alpha] \oplus \mathbb{Z}[2\alpha - \beta] \oplus \mathbb{Z}[\alpha - \gamma]\big) / \big(\mathbb{Z}[2\alpha - \beta] \oplus \mathbb{Z}[2(\alpha - \gamma)]\big) \approx \mathbb{Z} \oplus \mathbb{Z}_2.
\end{aligned}$$

Since X is a compact connected surface and $H_1(X) \approx H_1(\mathbb{R}P^2 \# \mathbb{R}P^2)$, by the classification theorem X is homeomorphic to $\mathbb{R}P^2 \# \mathbb{R}P^2$.

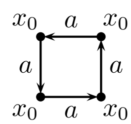
Bonus Problem

Let $D_4 = X_{aaaa}$ be the 4-fold dunce cap, i.e. the topological space represented by the diagram



(a) *Find a labeling scheme representing the universal cover $\tilde{D}_4$ for D_4. Describe $\tilde{D}_4$ as a subspace of $\mathbb{R}^3$ and the action of the group of covering transformations on $\tilde{D}_4$.*
(b) *Describe all (up to equivalence) covering maps with base D_4 (with path-connected covering spaces). For each covering space, find a labeling scheme representing it and describe the covering map.*

In this case all vertices of the square are mapped by the quotient projection map $q\colon P \longrightarrow D_4$ to the same point in D_4, which will be denoted by x_0:

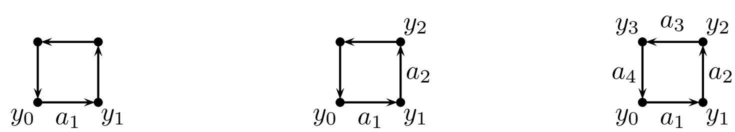


The image of the boundary of P consists of x_0 and a loop at x_0; it corresponds to a. The boundary of the square is wrapped around the loop four times in the same direction. Thus,

$$\pi_1(D_4, x_0) = \langle a | a^4 \rangle = \mathbb{Z}[a] / \mathbb{Z}[4a] \approx \mathbb{Z}_4;$$

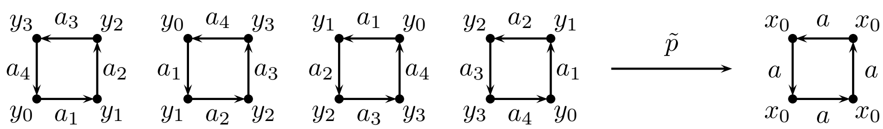
see the solution to Problem 8 for a similar computation described with more details.

(a) In order to construct $\tilde{p} : \tilde{D}_4 \longrightarrow D_4$, we need to turn the nontrivial elements of $\pi_1(D_4, x_0)$ into paths. Let's start with a square with the bottom left vertex labeled y_0 and the bottom edge labeled a_1. This vertex will be one of the preimages of x_0 under $\tilde{p}$. In the square for D_4, the path along the bottom edge is a loop in D_4 that generates $\pi_1(D_4, x_0)$. It lifts to a path in $\tilde{D}_4$ beginning at y_0, which is not a loop. This path corresponds to a_1. However, as this is not a loop in $\tilde{D}_4$, the bottom right vertex is no longer identified with the bottom left vertex. We label it y_1; it corresponds to a point of $\tilde{p}^{-1}(x_0)$ different from y_0:



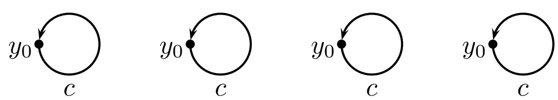
The lift of the loop a to $\tilde{D}_4$ beginning at a_1 corresponds to the right edge of the square. Since this path is different from a_1, the right edge is not identified with the bottom edge and we label it a_2. The path $a_1 a_2$ in $\tilde{D}_4$ is the lift of the loop $a^2 = aa$ beginning at y_0. Since a and a^2 are not the trivial elements in $\pi_1(D_4, x_0)$, the end point of $a_1 a_2$ is different from y_1 and y_0, respectively. We label it a_2. Proceeding in the same way, we find that the remaining vertex is again a new point and all four edges correspond to different paths in $\tilde{D}_4$; see the last diagram above.

The last square above, however, does not represent $\tilde{D}_4$. The restriction of q to a small neighborhood $\mathcal{U}$ of the bottom left vertex in P is a homeomorphism. The preimage of $q(\mathcal{U})$ under $\tilde{p}$ must consist of four spaces that look like $\mathcal{U}$, but with x_0 replaced by its preimages under $\tilde{p}$: y_0, y_1, y_2, and y_3. Applying the last paragraph's procedure to the squares with the bottom left vertex labeled y_1, y_2, and y_3, we obtain three new squares with labels:



The edges of the four squares must be identified as indicated. For example, the right edge in the first square and the bottom edge in the second square represent the lift of the loop a to $\tilde{D}_4$ that begins at y_1. So, they must be the same. The map $\tilde{p}$ takes each of the four squares on the left-hand side of the diagram above directly onto the square on the left, changing the labels a_k to a.

The above diagram is convenient for describing the covering map $\tilde{p}$. However, for describing the space $\tilde{D}_4$ it is more convenient to modify it. Since only the word $a_1a_2a_3a_4$ appears in the labeling scheme, we can replace it by a single letter, e.g. b. The space $\tilde{D}_4$ corresponds to



In other words, $\tilde{D}_4$ consists of 4 disks joined along the boundary circles. This space be embedded in $\mathbb{R}^3$. For example, it can be obtained by starting with two-spheres of radii 1 and 2 centered at the same point and joining them along the equators. The group of covering transforms, $\pi_1(D_4,x_0)=\mathbb{Z}_4$, acts by cyclicly permuting the disks and rotating the entire space. The generator of $\mathbb{Z}_4$ corresponding to the loop a in D_4 rotates the disks by one-quarter turn counterclockwise. In terms of the squares above, it takes the 1st square directly onto the 2nd, the 2nd onto the 3rd, the 3rd onto the 4th, and the 4th onto the 1st. Formally, if we identify $\pi_1(D_4,x_0)$ with the group $\{1,i,-1,-i\}$, then

$$\begin{aligned} \tilde{D} = \big(\{1,i,-1,-i\}\times B^2\big)/\sim, \qquad & (\lambda,z)\sim(\lambda',z)\ \ \forall\, z\in S^1,\quad \lambda,\lambda'\in\{1,i,-1,-i\}, \\ \text{and} \qquad & \lambda'\cdot(\lambda,z)=(\lambda'\lambda,\lambda'z)\ \ \forall\, z\in B^2,\quad \lambda,\lambda'\in\{1,i,-1,-i\}. \end{aligned}$$

(b) The coverings of D_4 correspond to the (conjugacy classes of) subgroups of $\pi_1(D_4,x_0)\approx\mathbb{Z}_4$. There are three such subgroups: the trivial subgroup, the entire group, and the two element subgroup

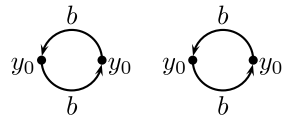
$$H=\{1,a^2\}=\{1,-1\}.$$

The trivial subgroup corresponds to the universal cover $\tilde{p}\colon \tilde{D}_4\longrightarrow D_4$ described in part (a). The entire group corresponds to the identity map $D_4\longrightarrow D_4$.

It remains to describe the covering $p: E\longrightarrow D_4$ corresponding to the subgroup H. We could proceed similarly to part (a) by constructing a labeling scheme directly. In this case, the loop a^2 lifts to a loop in E and we would end up with only two squares, two labels for vertices, and two labels for edges. However,

$$E=\tilde{D}_4/H$$

and we already know the universal covering $\tilde{D}_4$. Based on the description of $\tilde{D}_4$ and the group action in part (a), it follows that E can viewed as a two-sphere with opposite points on the equator identified. The corresponding labeling scheme is



The group of covering transformations, $\mathbb{Z}_4/\mathbb{Z}_2 \approx \mathbb{Z}_2$, acts by permuting the two hemispheres and rotating the entire sphere by one-quarter turn (which corresponds to half-turn for the circle $S^1/\mathbb{Z}_2 \subset E$). Formally,

$$\tilde{D} = \big(\{1,-1\} \times B^2\big)/\sim, \qquad (\lambda, z) \sim (\lambda', z) \ \ \forall\, z \in S^1, \ \ \lambda, \lambda' \in \{1,-1\},$$
$$\text{and} \qquad \lambda' \cdot (\lambda, z) = (\lambda'\lambda, \lambda' z) \ \ \forall\, z \in B^2, \ \ \lambda, \lambda' \in \{1,-1\}.$$

The map p takes each disk one-to-one to the square representing D_4, mapping each semicircle of the disk along two of the edges of the square. It is simplest to view this by using square diagrams: