Topology MAT 530 Homework 4

1. An *initial segment* in a well-ordered set X is a subset of the form

$$X_x = \{y \in X|\ y < x\}$$

where x is some element of X. Let $\mathcal{F}$ be a collection of well-ordered sets such that for any two sets $X, Y \in \mathcal{F}$ either X is an initial segment of Y or Y is an initial segment of X. Then the union of $\mathcal{F}$ is well-ordered. *We used this fact implicitly while proving the Zorn lemma.*

2. [Derivatives via the Nonstandard Analysis]. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Define $\hat{f} : hyper\mathbb{R} \to hyper\mathbb{R}$ as follows: if $x = (x_1, x_2, x_3, \dots)$, then $\hat{f}(x) = (f(x_1), f(x_2), f(x_3), \dots)$. Prove that the hyperreal number

$$\frac{f(x+\varepsilon) - f(x)}{\varepsilon},$$

where ε is any infinitesimal, differs from the derivative $f'(x)$ by some infinitesimal number. (Recall that an infinitesimal number ε is any positive hyperreal number such that $\varepsilon < a$ for any positive real a).

3. Prove that the direct product of an arbitrary collection of connected spaces is connected in the Tychonoff topology. *Hint:* Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be the product. Consider a subset $Y \subset X$ such that all projections of Y (except for finitely many of them) are points. Then Y is said to be of finite type. Prove that all subsets of finite type are connected, and that X can be obtained as the closure of the union of certain subsets of finite type with nonempty intersection.

4. Prove that the direct product of an arbitrary collection of Hausdorff spaces is Hausdorff in the Tychonoff topology.

5. Show that if X is regular, then every pair of points of X have neighborhoods whose closures are disjoint.

6. A closed subspace of a normal space is normal.

7. The product of two regular spaces is regular. *Hint:* this is proved in the textbook (not that I want you to copy it, but it can be used as a hint).