MAT 401: Undergraduate Seminar

Introduction to Enumerative Geometry

Fall 2008

More on Plane Conics

Theorem: For each integer i, with $0 \le i \le 5$, let $n_2(i)$ be the number of smooth plane conics that are tangent to i general lines and pass through $5-i$ general points in $\mathbb{C}P^2$. Then,

$$n_2(i) = n_2(5-i). \tag{*}$$

Corollary: The six numbers $n_2(i)$ are given by

i	0	1	2	3	4	5
$n_2(i)$	1	2	4	4	2	1

The numbers $n_2(0)$, $n_2(1)$, and $n_2(2)$ are computed in Chapter 2 of Katz's book; the identity $(*)$ then yields the remaining three numbers. The above theorem was the subject of the discussion on Thursday, 9/18; the aim of these notes is to sum up the full argument.

We will denote $[X]=[X_0,X_1,X_2]$ an arbitrary point of $\mathbb{C}P^2$ and by $[Y]=[Y_0,Y_1,Y_2]$ a specific point in $\mathbb{C}P^2$. A line in $\mathbb{C}P^2$ is the set of points $[X_0,X_1,X_2]$ in $\mathbb{C}P^2$ satisfying a linear equation

$$B_0X_0 + B_1X_1 + B_2X_2 = 0$$

for some $B=(B_0,B_1,B_2)\neq 0$ and thus corresponds to a point in the dual projective plane, $(\mathbb{C}P^2)^\vee \approx \mathbb{C}P^2$. Analogously, a point $[Y_0,Y_1,Y_2]$ in $\mathbb{C}P^2$ determines a line in $(\mathbb{C}P^2)^\vee$; it is the set of all points $[A_0,A_1,A_2]$ in $(\mathbb{C}P^2)^\vee$ such that

$$A_0Y_0 + A_1Y_1 + A_2Y_2 = 0.$$

Since a line in $(\mathbb{C}P^2)^\vee$ corresponds to a point in $\mathbb{C}P^2$, the dual of the dual of the original $\mathbb{C}P^2$ is the original $\mathbb{C}P^2$:

$$\big((\mathbb{C}P^2)^\vee\big)^\vee = \mathbb{C}P^2.$$

As we have seen previously, this duality between lines and points in the projective plane implies that the number of lines passing through two general points in $\mathbb{C}^2$ (or $\mathbb{C}P^2$) is the same as the number of points lying on two general lines in $\mathbb{C}^2$ (or $\mathbb{C}P^2 \approx (\mathbb{C}P^2)^\vee$).

A conic in $\mathbb{C}P^2$ is the zero set of a nonzero homogeneous polynomial $F(X_0,X_1,X_2)$. Any such polynomial is of the form

$$F_M(X_0,X_1,X_2) = X^tMX, \tag{**}$$

for some nonzero symmetric 3×3-matrix, where we view $X=(X_0,X_1,X_2)\in\mathbb{C}^3$ as a column vector. Let

$$C_M \equiv Z(F_M) \subset \mathbb{C}P^2$$

be the conic corresponding to a symmetric 3×3-matrix M. It can be of three possible shapes: smooth, union of two distinct lines, or a double line. These three possibilities are beautifully captured by the presentation (**):

Lemma 1: (1) If $\operatorname{rk} M=1$, C_M is a double line.
(2) If $\operatorname{rk} M=2$, C_M is a union of two distinct lines.
(3) If $\operatorname{rk} M=3$, C_M is a smooth conic.

This key technical observation was proved by Jonathan using algebraic computations only. Below we give a less direct argument.

- By the first part of Problem C in Problem Set III, if $[Y]\!\in\! C_M$ and $MY\!\neq\!0$, then $[Y]$ is a smooth point of C_M (this statement uses the assumption that M is symmetric). Thus, if the rank of M is 3, then every point of C_M is smooth (since the kernel of M is trivial) and thus C_M is a smooth conic.

- If the rank of M is two, M vanishes on a one-dimensional linear subspace of $\mathbb{C}^3$, corresponding to a singular point $[Y]\in C_M$, i.e. $MY=0$. Since M is symmetric, using the Gram-Schmidt diagonalization procedure we can also find $Y',Y''\in\mathbb{C}^3$ such that $\{Y,Y',Y''\}$ is a basis for $\mathbb{C}^3$,

$$Y'^t MY'=1, \qquad Y''^t MY''=1, \qquad Y'^t MY''=0;$$

  it is essential here that we are working with complex numbers. Then, $\{Y,Y'+\mathrm{i}Y''\}$ and $\{Y,Y'-\mathrm{i}Y''\}$ span 2 distinct two-dimensional linear subspaces of $\mathbb{C}^3$; their projectivizations are two distinct lines in $\mathbb{C}P^2$ intersecting at $[Y]\!\in\! C_M$. Furthermore, for all $s,t\!\in\!\mathbb{C}$

$$\begin{aligned}\big(sY+t(Y'+\mathrm{i}Y'')\big)^t M\big(sY+t(Y'+\mathrm{i}Y'')\big)&=0,\\ \big(sY+t(Y'-\mathrm{i}Y'')\big)^t M\big(sY+t(Y'-\mathrm{i}Y'')\big)&=0;\end{aligned}$$

  thus, these two lines must be contained in C_M. Since C_M is a conic (a degree 2 curve), it must then consist of these two lines only (each is a degree 1 curve).

- If the rank of M is one, M vanishes on a two-dimensional linear subspace of $\mathbb{C}^3$; its projectivization is a line L in $\mathbb{C}P^2$ on which F_M vanishes to second order, i.e. $C_M\!=\!2L$. Since M is symmetric of rank one, we can choose a basis $\{Y,Y',Y''\}$ for $\mathbb{C}^3$ such that $Y',Y''\!\in\!\ker M$ and thus $Y^tMY\!\neq\!0$. Then,

$$(aY+bY'+cY'')^t M(aY+bY'+cY'')=a^2\cdot Y^tMY;$$

  this expression vanishes only when $a\!=\!0$, i.e. $aY\!+\!bY'\!+\!cY''\!\in\!\ker M$. Thus, C_M has no points outside of L.

This completes the proof of Lemma 1.

If $C \subset \mathbb{C}P^2$ is a smooth conic, there is a well-defined tangent line $L_z(C)$ at each point $z \in C$; it corresponds to a point $\tau_C(z) \in (\mathbb{C}P^2)^\vee$. Thus, we obtain a map

$$\tau_C : C \longrightarrow (\mathbb{C}P^2)^\vee, \qquad z \longrightarrow \tau_C(z).$$

Lemma 2: The map τ_C is a homeomorphism onto a smooth conic $C^\vee$ in $(\mathbb{C}P^2)^\vee$ and $\tau_C^{-1} = \tau_{C^\vee}$. (In fact, τ_C is an analytic map, as is its inverse; such a map is called a biholomorphism).

Let M be a symmetric 3×3 matrix such that $C = C_M$. By Lemma 1, M is invertible, since C is smooth. By the last part of Problem C in Problem Set III, the tangent line to C_M at a point $Y \in C_M$ is given by $[MY] \in (\mathbb{C}P^2)^\vee$. Since $Y^t MY = 0$ for all $[Y] \in C_M$, it follows that

$$(MY)^t A^{-1}(MY) = 0 \ \ \forall\, [Y] \in C_M \qquad \Longrightarrow \qquad \tau_{C_M}(C_M) \subset C_{M^{-1}} \subset (\mathbb{C}P^2)^\vee.$$

On the other hand, if $[B] \in C_{M^{-1}}$, then $[M^{-1}B] \in C_M$. Thus,

$$\tau_{C_M} : C_M \longrightarrow C_M^\vee \equiv C_{M^{-1}}$$

is a bijection with inverse $\tau_{C_{M^{-1}}}$. Since M^{-1} has rank 3, by Lemma 1 the image of τ_{C_M} is a smooth conic. The map τ_{C_M} is continuous with respect to the quotient topology on $\mathbb{C}P^2$ because it is the composition of restrictions of the continuous maps

$$\mathbb{C}^3 - 0 \longrightarrow \mathbb{C}^3 - 0, \ \ X \longrightarrow MX, \qquad \mathbb{C}^3 - 0 \longrightarrow \mathbb{C}P^2, \ \ A \longrightarrow [A].$$

For the same reason, $\tau_{C_M}^{-1} = \tau_{C_{M^{-1}}}$ is also continuous. This concludes the proof of Lemma 2.

If C is a smooth conic which is tangent to a line L in $\mathbb{C}P^2$ at some point $[Y] \in C$, then the dual conic $C^\vee = \tau_C(C)$ passes through the point $L^\vee \in (\mathbb{C}P^2)^\vee$ corresponding to L, since $\tau_C([Y]) = L^\vee$. Conversely, suppose C is a smooth conic which passes through a point $p \in \mathbb{C}P^2$. Since $\tau_C^{-1} = \tau_{C^\vee}$, $p = \tau_{C^\vee}([B])$ for some $[B] \in C^\vee$ and thus $C^\vee$ is tangent at $[B]$ to the line $p^\vee \in (\mathbb{C}P^2)^\vee$ corresponding to $p \in \mathbb{C}P^2$.

We are now ready to prove the theorem. Choose i general lines L_j, $1 \le j \le i$, and $5-i$ general points p_j, $1 \le j \le 5-i$, in $\mathbb{C}P^2$. They correspond to i general *points* $L_j^\vee$, $1 \le j \le i$, and $5-i$ general *lines* $p_j^\vee$, $1 \le j \le 5-i$, in $(\mathbb{C}P^2)^\vee$. If C is a smooth conic in $\mathbb{C}P^2$ which is tangent to the i lines L_j and passes through the $5-i$ points p_j, then by the previous paragraph $C^\vee \subset (\mathbb{C}P^2)^\vee$ is a smooth conic which passes through the i points $L_j^\vee$ and is tangent to the $5-i$ lines $p_j^\vee$. Conversely, if $C^\vee \subset (\mathbb{C}P^2)^\vee$ is a smooth conic which passes through the i points $L_j^\vee$ and is tangent to the $5-i$ lines $p_j^\vee$, then $C = (C^\vee)^\vee \subset \mathbb{C}P^2$ is a smooth conic which is tangent to the i lines L_j and passes through the $5-i$ points p_j. Thus, we have established a bijection between the set of smooth conics in $\mathbb{C}P^2$ which are tangent to the i lines L_j and pass through the $5-i$ points p_j and the set of smooth conics in $(\mathbb{C}P^2)^\vee \approx \mathbb{C}P^2$ which pass through the i points $L_j^\vee$ and are tangent to the $5-i$ lines $p_j^\vee$. By definition, the cardinality of the first set is $n_2(i)$, while the cardinality of the second set is $n_2(5-i)$; this proves the identity $(*)$.