MAT 320: Introduction to Analysis, Spring 2018
Homework Assignment 9

Please read Ross's Sections 23-26 thoroughly and before starting on the problem set below.
Optional supplementary reading: pp143-158 of Rudin's book

Problem Set 9 (**due at the start of recitation on Wednesday, 4/18**): 23.2*, Problems P-S (below and next page); *answers only on 23.2 (8 in total)

Problem P

Suppose the radius of convergence of $\sum a_n z^n$ is $R \in \mathbb{R}^+$.

(a) Suppose $a_n \geq 0$ and the series converges for $z = R$. Show that the series converges for every $z \in \mathbb{C}$ with $|z| \leq R$.

(b) Give an example of a series that converges at $z = 1$ and diverges at $z = -1$.

Problem R

Let X be a set, (Y, d_Y) be a complete metric space, and $f_n : X \longrightarrow Y$ be a uniformly Cauchy sequence of functions.

(a) Show that the sequence f_n converges uniformly to some function $f\colon X \longrightarrow Y$.

(b) Suppose in addition that d_X is a metric on X and each f_n is continuous function from (X, d_X) to (Y, d_Y). Show that the sequence f_n converges uniformly to some continuous function f from (X, d_X) to (Y, d_Y).

Hint: A special case of (a) is HW6-13.3c. This problem is also related to Theorem 24.3, a more general version of which was proved in class, and to Theorem 25.4.

Problem Q

Let X be a set and (Y, d_Y) be a metric space. A function $f\colon X \longrightarrow Y$ is called bounded if $f(X) \subset Y$ is a bounded subset (contained in some $B_R^Y(y)$). Denote by $\mathfrak{B}(X, Y)$ the set of bounded functions from X to Y. Define

$$d\colon \mathfrak{B}(X,Y)^2 \longrightarrow \mathbb{R}, \qquad d(f,g) = \sup_{x \in X} d_Y\big(f(x), g(x)\big).$$

(a) Show that d is well-defined (takes values in $\mathbb{R}$) and is a metric on $\mathfrak{B}(X, Y)$.

(b) Show that d is a complete metric on $\mathfrak{B}(X, Y)$ if d_Y is a complete metric on Y.

Suppose in addition that d_X is a metric on X. Let $\mathfrak{CB}(X, Y) \subset \mathfrak{B}(X, Y)$ be the subset of continuous bounded functions from (X, d_X) to (Y, d_Y).

(c) Show that (the restriction of) the metric d on $\mathfrak{CB}(X, Y)$ is complete if d_Y is a complete metric on Y.

(d) Suppose $f_n \in \mathfrak{CB}(X, Y)$ is a sequence of uniformly continuous functions which converges to a uniformly continuous function $f \in \mathfrak{CB}(X, Y)$ with respect to the metric d. Show that the sequence of functions f_n is equicontinuous, i.e. for every $\epsilon \in \mathbb{R}^+$ there exists $\delta \in \mathbb{R}^+$ such that

$$d_Y\big(f_n(x), f_n(x')\big) < \epsilon \qquad \forall\, n \in \mathbb{N},\ x, x' \in X \text{ s.t. } d_X(x, x') < \delta;$$

thus, δ depends only on ϵ, not on n.

Problem S

Power series are typically used to "break" a function into a sequence of numbers (the Taylor and Fourier coefficients of the function). However, sometimes it is useful to go in the opposite direction, assembling a sequence of numbers into a function.

Let f_n be the n-th Fibonacci number defined recursively by

$$f_0 = 0, \qquad f_1 = 1, \qquad f_{n+2} = f_n + f_{n+1} \quad \forall\, n \in \mathbb{Z}^{\geq 0}.$$

Let

$$A_n = \sum_{k=1}^{k=n} k = 1+2+\ldots+n, \quad B_n = \sum_{k=1}^{k=n} k^2 = 1^2+2^2+\ldots+n^2 \qquad \forall\, n \in \mathbb{Z}^{\geq 0}.$$

(a) Give a recursive definition of the numbers A_n, B_n with $n \geq 0$.

(b) Use mathematical induction and only the recursive definitions of f_n, A_n, B_n to show that $f_n, A_n, B_n \leq 5^n$ for all $n \geq 0$.

(c) Use the *Absolute Convergence* and *Comparison* Tests and only part (b) to show that the power series

$$f(x) = \sum_{n=0}^{\infty} f_n x^n, \qquad A(x) = \sum_{n=0}^{\infty} A_n x^n, \qquad B(x) = \sum_{n=0}^{\infty} B_n x^n,$$

converge if $|x| < 1/6$ (and thus determine smooth functions near $x=0$).

(d) Using only the recursive definitions of f_n, A_n, B_n, show that

$$f(x) = x + xf(x) + x^2 f(x), \quad A(x) = xA(x) + \frac{x}{(1-x)^2}, \quad B(x) = xB(x) + \frac{x}{(1-x)^2} + \frac{2x^2}{(1-x)^3}.$$

Hint: You'll need to use identities such as the following:

$$\frac{1}{(1-x)^3} = \frac{1}{2}\left(\frac{1}{1-x}\right)'' = \frac{1}{2}\left(\sum_{n=0}^{\infty} x^n\right)'' = \frac{1}{2}\sum_{n=0}^{\infty} n(n-1)x^{n-2}.$$

(e) Using only part (d), express f_n, A_n, and B_n explicitly in terms of n.
Hint: use (d) to solve for f, A, and B and expand them into Taylor series around $x=0$ (partial fractions might help in the case of f); compare the result with the definitions of f, A, and B in (c).

Note: For f_n, you should end up with a formula involving two square roots. There is a much simpler way of finding an explicit formula for A_n; so you can check your answer, but please deduce this formula from (b). The answer for B_n can be confirmed using induction (or google).