MAT 319/320: Basics of Analysis, Spring 2018
Homework Assignment 5

Please read Sections 14-16 from Ross's textbook thoroughly before starting on the problem set below.

Optional supplementary reading: pp58(bottom)–69(top) from Rudin's book

Problem Set 5 (**due at the start of recitation on Wednesday, 2/28**:
14.2*, 14.6, 14.10, 14.12, 14.13b, 15.3, 15.6, 16.10, Problem C (below)
*This (14.2) is *answer only*

Please write your solutions legibly; the grader may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem C

(a) Let $z\in\mathbb{C}$ with $z\neq m\pi$ for any $m\in\mathbb{Z}-\{0\}$. Show that the series

$$g(z) \equiv \sum_{n=1}^{\infty}\left(\frac{1}{z-n\pi}+\frac{1}{z+n\pi}\right)$$

converges. *Hint:* combine the fractions and use the *Absolute Convergence Test.*

(b) Let $\epsilon\in\mathbb{R}^+$. Show that there exists $\delta\in\mathbb{R}^+$ such that $|g(z)|<\epsilon$ whenever $|z|<\delta$.
Hint: find $\delta<1$ using your work in (a).

(c) By (a), the function

$$f(z) \equiv \frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z-n\pi}+\frac{1}{z+n\pi}\right)$$

is well-defined for $z\in\mathbb{C}$ with $z\neq m\pi$ for any $m\in\mathbb{Z}$. Show that

$$\lim_{z\longrightarrow 0} zf(z)=1, \qquad f(-z)=-f(z), \qquad f(z+\pi)=f(z), \qquad f(\pi/2)=0, \tag{1}$$

with the middle identities holding whenever either side is defined ($z\neq m\pi$ for $m\in\mathbb{Z}$).
Hint: use partial sums for the third equality; the second one is easy; the fourth follows from the second and third. The formal meaning of the first equality will be defined in Chapter 3, but use (b) to justify it as in MAT 125/131).

(d) Show that the function $h(z)=\cot z$ also satisfies all equalities in (1).

(e) Along with a little bit of MAT 342, this gives $f(z)=\cot z$. Differentiate both sides of this equality (as in one-variable calculus), add $1/z^2$ to both, and take the limit of both sides as $z\longrightarrow 0$ to obtain

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}\,.$$