MAT 320: Introduction to Analysis, Spring 2019
Homework Assignment 8

Please study Section 21,22 of Ross's textbook thoroughly and read through Sections 17-20 before starting on the problem set below.

Optional supplementary reading: Chapter 4 of Rudin's book

Problem Set 8 (**before the start of recitation on Wednesday, 4/10**):
21.4, Problems M-O (below), 17.14

Problem M

Let $f\colon [0,1] \longrightarrow [0,1]$ be a continuous function (with respect to the standard metric). Show that f has a fixed point, i.e. $f(x^*)=x^*$ for some $x^* \in [0,1]$.

Problem N

Let (X, d_X) and (Y, d_Y) be metric spaces and $f\colon X \longrightarrow Y$ be a map. For each $x \in X$, define

$$\omega_f(x) = \inf \big\{ \sup \big\{ d_Y(f(x'), f(x'')) \colon x', x'' \in U \big\} \colon U \subset X \text{ open}, x \in U \big\}.$$

(a) Show that f is continuous at x if and only if $\omega_f(x)=0$.

(b) Show that for every $\epsilon>0$ the set $U_f(\epsilon) \equiv \{x \in X \colon \omega_f(x) < \epsilon\}$ is open.

(c) Show that there exists no function $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ which is continuous at all points of $\mathbb{Q}^n \subset \mathbb{R}^n$ and discontinuous at all points of $\mathbb{R}^n - \mathbb{Q}^n$.

Hint: Example 8 in Section 21 and Problem I on HW7 might be helpful. Giving a name to the sup above, such as $\omega_{f;U}$, might be convenient.

Problem O

Let (X, d) be a metric space and $A \subset X$ be a non-empty subset.

(a) Show that the map

$$d_A\colon X \longrightarrow \mathbb{R}^{\geq 0}, \qquad d_A(x) = \inf \big\{ d(a,x) \colon a \in A \big\},$$

is well-defined and continuous. Furthermore, $d_A(x)=0$ for all $x \in A$.

(b) Show that $d_A(x) \neq 0$ for all $x \notin A$ if and only if $A \subset X$ is closed.

(c) Let $B \subset X$ be another subset, which is disjoint from A. Show that

$$d(A,B) \equiv \inf \big\{ d(a,b) \colon a \in A,\, b \in B \big\} > 0$$

if A is closed and B is compact.

(d) Give an example showing that the above conclusion may fail if A is not assumed to be closed (but B is compact) and another example showing that the above conclusion may fail if A and B are closed, but neither is compact.