MAT 127 LECTURE OUTLINE WEEK 8

These lecture notes are meant to complement what is found in the textbook, to explain the same material in a slightly different way. My aim is to keep these relatively concise, while pointing you to the textbook for more details as needed.

Goal: We continue our study of infinite series. In practice, it is often difficult or impossible to evaluate a series exactly. Instead, the first step is to determine whether a series converges or diverges. If it converges, one can then compute a numerical approximation.

(1) Recall that an **infinite series** is an infinite sum, written as

$$\sum_{n=1}^{\infty} a_n$$

or

$$a_1 + a_2 + a_3 + \cdots .$$

In simple cases, a series can be evaluated exactly. This is the case for geometric series, which we covered last week. Usually, however, this is difficult or impossible.

For this reason, the analysis of a particular series typically has two parts. First, we need to decide whether the series converges (i.e., the sequence of partial sums has a limit) or diverges. There are various **tests for convergence** that we will begin to go over this week. Second, if the series converges, then we can find some numerical approximation for the value.

(2) Before going further, there is one more situation worth mentioning when a series can be evaluated exactly: **telescoping series**. This is a series where the majority of the terms in each partial sum cancel, leaving a finite number of terms that can be evaluated directly. Let's do an example:

Example. Find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$.

Since we a rational function as the summand, we try using partial fraction decomposition. We can write the series as

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} \frac{A}{n} + \frac{B}{n+2}.$$

We cross multiply to find $A = 1/2$ and $B = -1/2$. So we have

$$\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2}.$$

Now we get to the "telescoping" part: for each k, the k-th partial sum is

$$S_k = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{k} - \frac{1}{k+2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{k+1} - \frac{1}{k+2}\right).$$

We then have

$$\sum_{n=1}^{\infty} \frac{1}{n^2+2n} = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{k+1} - \frac{1}{k+2}\right) = \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2}\right) = \frac{3}{4}.$$

(3) We're now ready to begin our study of **tests of convergence**. The first and most basic criterion is called the **divergence test**. It states the following:

> Divergence test. If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Equivalently, if $\lim_{n\to\infty} a_n$ does not exist or is non-zero, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

The idea is that if $\sum_{n=1}^{\infty} a_n$ converges, then the partial sums S_k approach a limit. But a_n is the difference $S_k - S_{k-1}$, which necessarily approaches 0.

Example. From this test, we can see that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n+1}$ diverges, since $\dfrac{n-1}{n+1} \to 1 \neq 0$ as $n \to \infty$.

(4) The next test is called the **integral test** and is based on comparing a sum with a matching integral. Let's say that $a_n = f(n)$ for some function $f(x)$ defined on the positive real numbers, which is usually the case. The integral test states that, under mild conditions, the sum $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\displaystyle\int_a^{\infty} f(x)\,dx$ exists for some $a > 0$. The precise statement is Theorem 5.9 in the book:

> Integral test. Suppose $\displaystyle\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n. Suppose there exists a function $f(x)$ defined on an interval $[N, \infty)$ satisfying the following: (a) f is continuous; (b) f is decreasing; (c) $f(n) = a_n$ for all integers $n \geq N$. Then $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\int_N^{\infty} f(x)\,dx$ either both converge or both diverge.

Note that the actual value of the sum and the integral will usually be different. Also, we started the sum at $n = 1$, but the starting value of n doesn't matter, since the convergence of $\displaystyle\sum_{n=1}^{\infty} a_n$ depends only on what happens to a_n as n gets arbitrarily large.

(5) For our first example, we can show in a different way that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. This follows from the integral test by taking

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln(x)|_1^b = \lim_{b\to\infty} \ln(b) = \infty.$$

Take a moment to check that all the conditions are satisfied: $1/x$ is positive, continuous and decreasing. When doing homework or taking the midterm, make sure to justify why the test applies as part of your answer.

(6) More generally, we can consider the p**-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

where p is some real number. Note that if $p \leq 0$, then the p-series diverges by the divergence test. So we're left with the case that $p > 0$. If $p = 1$, then we have the harmonic series, which diverges. Otherwise, we can integrate as follows:

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx = \lim_{b\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^b = \lim_{b\to\infty} \left(\frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1}\right).$$

If $p < 1$, then this limit is infinite and so the corresponding integral/series diverges. If $p > 1$, then the limit exists, so the series converges. This is a good fact to commit to memory:

$$\boxed{\text{The } p\text{-series } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if and only if } p > 1.}$$