MAT 127 LECTURE OUTLINE WEEK 4-5

These lecture notes are meant to complement what is found in the textbook, to explain the same material in a slightly different way. My aim is to keep these relatively concise, while pointing you to the textbook for more details as needed.

Goal: We have two separate topics to cover: **logistic growth** and **second-order homogeneous constant coefficient linear equations**.

(1) We will continue with the theme of mathematical modelling, which has been a consistent part of the this course and which we discussed in more depth last week. Here, we will look at basic models of population growth. In this context, our variable is t to represent "time", and our function is denoted by $P(t)$ to represent "population". Like all models, the ones we consider here are simplifications or idealizations of reality.

(2) Our starting point is a review of exponential growth. This corresponds to a population that grows at a fixed rate proportional to the current population without any constraint to limit growth. For example, you might think of a population of bacteria where each bacteria divides every three hours. Note, however, that while the actual population must be an integer at each point in time, we use a <u>continuous</u> function as a model for population. The equation for exponential growth is

$$P(t) = P_0 e^{rt},$$

where $P_0 \geq 0$ is the initial population size and r is the **growth rate**. In this context, we require that $r > 0$; if $r < 0$, then $P(t)$ instead represents exponential decay. The exponential growth equation is most likely one you would have seen in precalculus or elsewhere. In the context of our class, we can think of it as a solution to a particular initial value problem:

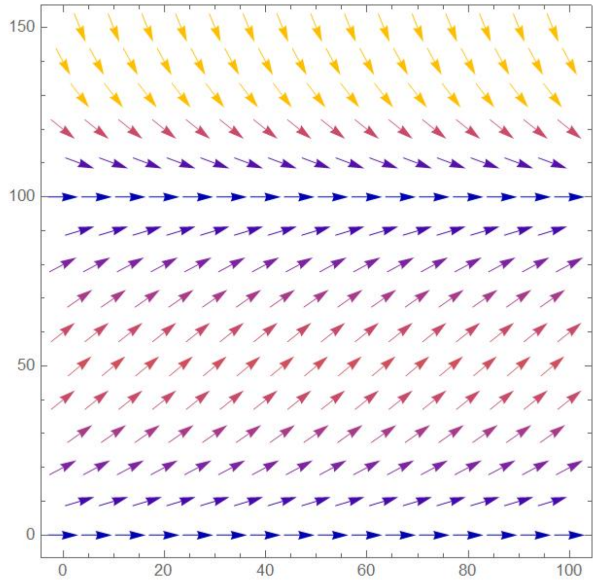
$$\frac{dP}{dt} = rP, \quad P(0) = P_0.$$

(3) As a mathematical model, exponential growth is fairly limited, since it doesn't account for limitation on space and resources, and so forth. The next level of model for population growth is to consider a **carrying capacity** for an organism in a given environment: the maximum population that can be sustained indefinitely. The **logistic differential equation** is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right),$$

where r is the growth rate and K is the carry capacity of the environment. It looks like the equation for exponential growth but with an extra factor of $(1 - P/K)$.

(4) To get a feel for the equation, let's plot a typical direction field, taking $K = 100$:



The qualitative features should be evident: if P is small, then the population increases in a way very similar to exponential growth. Once P gets close to 100, the rate of growth tails off, and P approaches 100 in the limit. If the initial population is larger than 100, then the population declines until 100 is reached. There are two equilibrium solutions at 0 and 100. What is their asymptotic stability? Can you give an interpretation of this?

Certainly, the logistic growth model is an improvement over the exponential growth model. However, you might think about its limitations. For example, it doesn't directly account for interactions between populations of different organisms.

(5) The logistic growth equation is autonomous, since it does not depend on time. For this reason, it can analyzed using a **phase line**: to represent on a vertical line the zones where P is either increasing, decreasing, or flat.

(6) The logistic equation can be solved using separation of variables, as we've practiced in the previous section. We set this up as

$$\int \frac{1}{P(1-P/K)}\,dP = \int r\,dt.$$

The left-hand side can be written using partial fractions and then integrated, giving:

$$\int \frac{1}{P} + \frac{1}{K-P}\,dP = \ln|P| - \ln|K-P| + C = \ln\left|\frac{P}{K-P}\right| + C.$$

The right hand side integrates to $rt + C$. Together (and consolidating the constants of integration into a single constant C), this gives

$$\ln\left|\frac{P}{K-P}\right| = rt + C.$$

Exponentiating and removing the absolute value signs gives

(1) $$\frac{P}{K-P} = C_1 e^{rt},$$

where C_1 is a real constant. With some algebra, we can solve for P to get

$$P(t) = \frac{C_1 K e^{rt}}{1 + C_1 e^{rt}}$$

as the general solution. We can now use the initial value $P(0) = P_0$ to determine the constant C_1. Here, it's more convenient to use the equation (1) above. This gives $C_1 = P_0/(K - P_0)$. With some more algebra, we get

$$P(t) = \frac{P_0 K e^{rt}}{K - P_0 + P_0 e^{rt}}.$$

(7) Here's a typical problem, taken from the textbook: Suppose that a butterfly sanctuary has a capacity of 2000 butterflies and an initial population of 400. After two months, you observe that the population is at 800. Use the logarithmic growth model to predict how long it will take to reach a population of 1500 butterflies.

We are given that $K = 2000$, but we are not given the growth rate r. However, according to (1), we have

$$\frac{800}{2000 - 800} = \frac{400}{2000 - 400} e^{r \cdot 2},$$

which simplifies to

$$\frac{2}{3} = \frac{1}{4} e^{2r}.$$

So $r = \ln(8/3)/2$. We can now use (1) again to solve the problem: the time t with population $P(t) = 1500$ occurs when

$$3 = \frac{1500}{2000 - 1500} = \frac{1}{4} e^{\ln(8/3)t/2}.$$

Solving for t gives

$$t = \frac{2\ln(12)}{\ln(8/3)} \approx 5.07 \text{ months}.$$

(8) Now we switch to the second topic: **2nd order homogeneous constant coefficient linear equations**. This is a lengthy way to refer to differential equations of the form

$$y'' + by' + cy = 0,$$

where $y = y(x)$ and b, c are real numbers. Let's elaborate on the terminology. The word "homogeneous" here means that the right-hand side of the equation is 0. A "linear" equation would allow for something of the form $y'' + b(x)y' + c(x)y = 0$, i.e., the coefficients b and c are actually functions of x. But here we have specified that the equation has "constant coefficients", so b and c are just constants.

(9) The DE notes posted on the course website already give an overview of this topic. Instead of duplicating what is already written, I'll just make a few supplementary remarks. First is a word regarding applications: these equations commonly show up when describing an object in motion subject to some force (recall that a force causes acceleration on the object, which corresponds to a second derivative). We've already seen one simple example of this in the form of *projectile motion*, where the force is due to gravity. A second example is an *oscillating spring*. Imagine a spring attached to the ceiling and hanging down vertically and a weight attached to the bottom of the spring. The spring is subject to a *restoring force* that acts on the weight if it

is pulled down or pushed up from equilibrium position. If you pull the weight away from equilibrium position and then let go, then this force will cause the weight to oscillate up and down. This type of system is described by an equation of the form $y'' + by' + cy = 0$. We will focus our effort in this lesson on the mathematics itself rather than the applications, but this is a good example to keep in mind.

(10) The main mathematical property of linear homogeneous equations is usually called the **principle of superposition**: that if y_1 and y_2 solve such an equation, then any **linear combination** $C_1y_1 + C_2y_2$ is also a solution. It's good to recall here that the general solution of a second-order equation contains two unknown constants C_1 and C_2. Thus, if we can find two distinct solutions $y_1(x)$ and $y_2(x)$ to a second order homogeneous linear equation, then the general solution is

$$y(x) = C_1y_1(x) + C_2y_2(x).$$

(11) The DE notes on the webpage describe a procedure of guessing a solution has the form $y(x) = e^{\lambda x}$. Plugging this into the equation $y'' + by' + cy = 0$ leads to the equation

$$\lambda^2 + a\lambda + b = 0, \tag{2}$$

which can be solved either by factoring or using the quadratic equation. Note that the solutions to (2) may very well be **complex numbers**, i.e., numbers containing the imaginary unit $i = \sqrt{-1}$. This is a topic you should review if it is unfamiliar. There are three cases that can occur:

- The equation (2) has two real solutions $\lambda = a_1, a_2$.
  Then the general solution is
  $$y(x) = C_1e^{a_1x} + C_2e^{a_2x}.$$
- The equation (2) has two complex solutions $\lambda = a \pm bi$. Note that these complex numbers must be complex conjugates, i.e., a pair $a \pm bi$. It is mathematically correct to say that the general solution is
  $$y(x) = C_1e^{(a+bi)x} + C_2e^{(a-bi)x}.$$
  However, it is more satisfying to give a solution in which the imaginary unit i does not appear. This can be done using some cleverness and **Euler's formula**:
  $$e^{ia} = \cos(a) + i\sin(a).$$
  See the DE notes on the webpage for details. The result is that we can write the general solution in the form
  $$y(x) = C_1e^{ax}\cos(bx) + C_2e^{ax}\sin(bx).$$
- The equation (2) has a single solution $\lambda = a$, which is necessarily real. Then the general solution is
  $$y(x) = C_1e^{ax} + C_2xe^{ax}.$$

  As always, if the problem has initial values, we can use them to determine the coefficients C_1, C_2 for the particular solution. For a second order equation, there will be two initial values, and then we must solve a system of two linear equations in two variables. Section 8 of the DE notes on the webpage give an example of this.