MAT 127 LECTURE OUTLINE WEEK 3

These lecture notes are meant to complement what is found in the textbook, to explain the same material in a slightly different way. My aim is to keep these relatively concise, while pointing you to the textbook for more details as needed.

Goal: We will learn to solve differential equations by **separation of variables**. In the process, we will also review techniques of integration such as integration by partial fraction decomposition.

(1) We will finally learn our first method for solving a differential equation. It applies whenever a differential equation can be written in the form $y' = f(x)g(y)$ (or $\frac{dy}{dx} = f(x)g(y)$), or more suggestively in the form

$$\frac{1}{g(y)}dy = f(x)dx.$$

Notice that the variable y occurs only on the left-hand side of the equation, while the variable x occurs only on the right-hand side. A differential equation of this type is called **separable**, since the equation can be separated into a function of x and a function of y.

Some examples are $y' = (x-3)(y^2+1)$, $y' = \sin(x) + \sqrt{x}$ and $y' = y^2$. It is also evident that every <u>autonomous</u> first-order differential equation (such as the third example) is separable.

(2) The basic idea for solving a separable equation is simple: put the equation in the form $(1/g(y))dy = f(x)dx$ and integrate both sides (as <u>indefinite</u> integrals—make sure to include the additive constant C!):

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx.$$

For the example $y' = (x-3)(y^+1)$, we have

$$\int \frac{1}{y^2+1}\,dy = \int x - 3\,dx,$$

which gives

$$\arctan(y) = x^2 - 3x + C.$$

If possible, we then solve for y in terms of x. [This is not always possible, so we end up with an <u>implicit</u> function $y = y(x)$.] In this example, we have

$$y = \tan(x^2 - 3x) + C.$$

At this point, if we are given an initial value, we can use it to determine the constant C.

(3) Let's return to the Newton's law of cooling example from last week. A typical equation is

$$y' = -.2(y - 70).$$

We have then

$$\begin{aligned} &\int \frac{1}{y-70}\,dy = -.2\int dx \\ \implies\quad &\ln|y-70| = -.2x + C \\ \implies\quad &|y-70| = e^{-.2x+C}. \end{aligned}$$

Don't forget the absolute value sign around "$y - 70$". The absolute value sign can be removed by writing the solution in the form

$$y = 70 + ce^{-.2x},$$

where $c = \pm e^C$. Since C is an undetermined constant, it's fine to replace C with some other undetermined constant that achieves the same thing like the c here.

Say we're given an initial value $y(0) = 55$. Then we have $55 = 70 + c \cdot 1$, so that $c = -15$. So the particular solution is

$$y = 70 - 15e^{-.2x}.$$

(4) Let's return to the subject of <u>mathematical modelling</u>: creating a mathematical model to represent a physical situation. We've already seen a couple examples: temperature and projectile motion. In the homework you'll do projectile motion <u>with air resistance</u>, which we've generally neglected in calculus up to this point.

A type of problem we'll look at now is a mixing problem; see also Example 4.12 in the book. Imagine a tank containing a fluid with a certain concentration of salt. Let's say the tank initially contains 100 liters of water with 6 kilograms of salt. We open a plug and let the salty water drain out of the tank at a rate of 2 liters per minute. At the same time, we start adding water at the same rate of 2 liters per minute with a concentration of .2 kilograms per liter. Thus the volume of water in the tank remains constant. The problem is to find the amount of salt in the tank as a function of time.

We can frame this problem as a differential equation. Let $u(t)$ denote the amount of salt in the tank as a function of time. The function $u(t)$ is controlled by the inflow and outflow of salt. Each minute, the tank loses $1/50$th of its salt. On the other hand, it gains a constant .4 kilograms of salt per minute.

This gives $u'(t) = .4 - u(t)/50$ or $u' = (20-u)/50$. We can now solve this differential equation similarly to the Newton's law of cooling example:

$$\begin{aligned} &\int \frac{1}{20-u}\,du = \int \frac{1}{50}\,dt \\ \implies\quad &-\ln|20-u| = \frac{t}{50} + C \\ \implies\quad &|20-u| = e^{-t/50-C} = e^{-C}\cdot e^{-t/50} \\ \implies\quad &u = 20 - C_1 e^{-t/50} \ \text{(where } C_1 = \pm e^{-C}\text{)}. \end{aligned}$$

This is the general solution. For the particular solution to our initial value problem, we have $u(0) = 6 = 20 - C_1 e^0 = 20 - C_1$, so that $C_1 = 14$. So the solution is $u(t) = 20 - 14e^{-t/50}$.

(5) Any separable equation can be solved this way, though often the difficult step is to actually do the integrals. All the techniques of integration you've learned before will

come in handy. For example, integration by partial fraction decomposition often comes up. For example, let's do the equation

$$y' = (2x+3)(y^2-4).$$

This becomes

$$\int \frac{1}{y^2-4}\,dy = \int (2x+3)\,dx.$$

For the left-hand side, we write it in the form

$$\int \frac{1}{y^2-4}\,dy = \int \frac{1}{(y-2)(y+2)}\,dy = \int \frac{A}{y-2} + \frac{B}{y+2}\,dy$$

for some coefficients A, B yet to be determined. But we see that

$$\frac{A(y+2)+B(y-2)}{(y-2)(y+2)} = \frac{1}{(y-2)(y+2)},$$

and so $A(y+2)+B(y-2) = 1$. Plug in $y = 2$ to get $A \cdot 4 + 0 = 1$, so $A = 1/4$. Plug in $y = -2$ to get $0 + b \cdot (-4) = 1$, so $B = -1/4$. Evaluating the integrals above, we have

$$\frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2| = x^2+3x+C.$$

This can be rewritten as

$$\ln\left|\frac{y-2}{y+2}\right| = 4x^2+12x+C.$$

So

$$\frac{y-2}{y+2} = C_1 e^{4x^2+12x}.$$

Isolating the y, we get

$$y = \frac{2+2C_1e^{4x^2+12x}}{1-C_1e^{4x^2+12x}}.$$

This is the general solution.