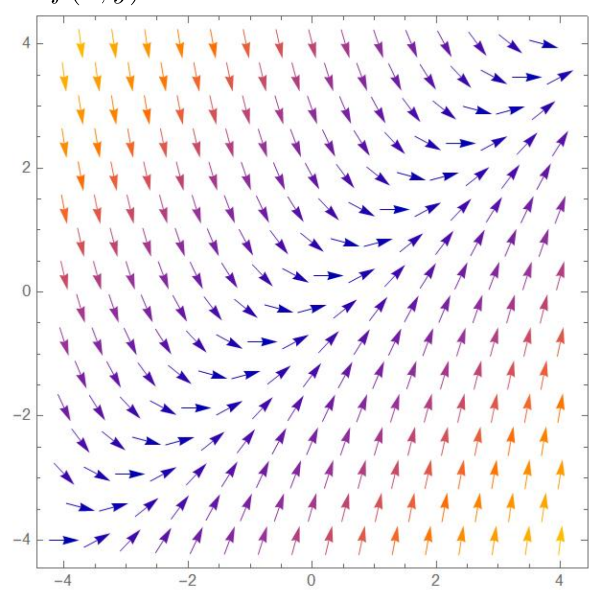
MAT 127 LECTURE OUTLINE WEEK 2

These lecture notes are meant to complement what is found in the textbook, to explain the same material in a slightly different way. My aim is to keep these relatively concise, while pointing you to the textbook for more details as needed.

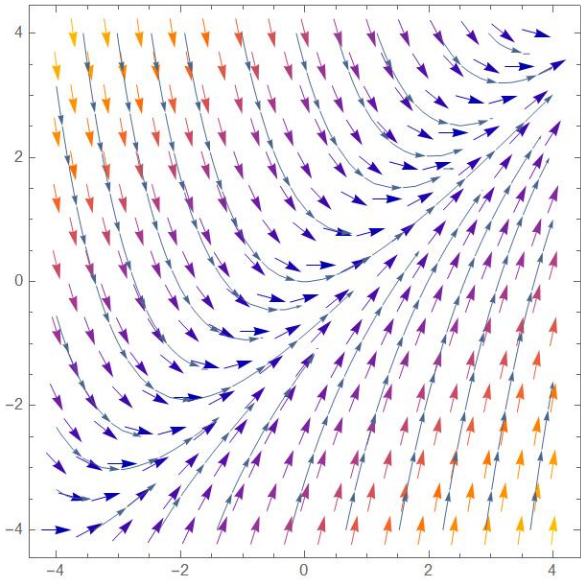
Goal: We cover the two related topics of **direction fields** and **Euler's method** in order to study first-order differential equations.

(1) This week, we will investigate first-order differential equations, which typically can be written in the form
$$y' = f(x, y)$$
for some function $f(x, y)$, where x is the independent variable and $y = y(x)$. A typical example is
$$y' = x - y.$$
(Sometimes, t is used instead of x for the independent variable.) Later, beginning in Week 3, we will learn how to find exact solutions to these equations. For now, we will try to understand these solutions qualitatively (understanding the main features and behavior) and numerically (using a computer or calculator to approximate them).

(2) The main tool we use to understand solutions qualitatively is a **direction field**, also called a **slope field**. To make a direction field, draw a short arrow at each point whose slope is equal to $f(x, y)$. Here is a direction field for the equation $y' = x - y$:



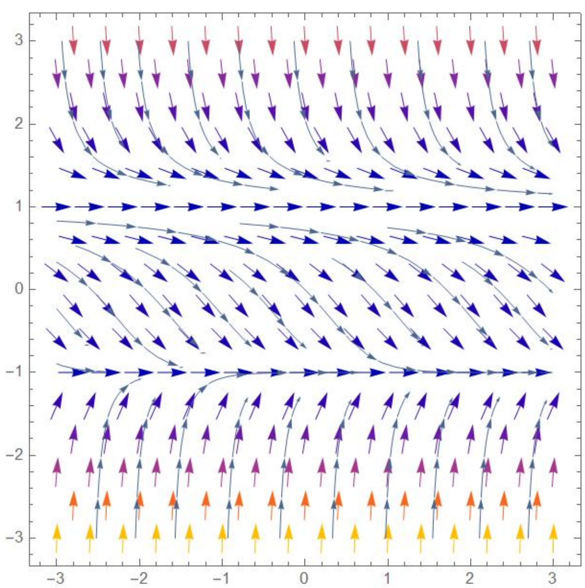
Notice the the equation $y' = f(x, y)$ is a condition on the slope of $y(x)$ at each point. So a solution to the differential equation is "steered" by the directional field. That is, the solution must run parallel to the arrows you've drawn. This gives a visual way to imagine the solutions to the differential equation. A few solutions to the differential equation $y' = x - y$ are shown here:



As we discussed last week, a unique solution is obtained once an initial value $y_0 = y(x_0)$ is chosen.

From the direction field, you can get a good idea of the general shape of a solution. In particular, you can often get a feel for the **asymptotic** or long-term behavior of the solution. For example, a solution will often converge to a certain limiting value as $x \to \infty$.

(3) One useful strategy is to identify the **equilibrium solutions**, or constant solutions, to the differential equation. These are solutions of the form $y = k$ for some real number k. Since they are constant, they must satisfy $y' = f(x, k) = 0$. Graphically, these solutions are horizontal lines. Try to identify the equilibrium solutions in the direction field here:



It is evident that these are $y = 1$ and $y = -1$. Next, we can classify each equilibrium solution $y = k$ as one of three types: **asymptotically stable**, **asymptotically unstable**, and **asymptotically semistable**. Stable means that all solutions with initial value sufficiently close to k will have $y = k$ as a limit as $x \to \infty$. Unstable means that all solutions with initial value near $y = k$ (but not equal to k) do not have $y - k$ as a limit as $x \to \infty$. Semistable means that neither of the previous two are true; that is, some solutions limit to $y = k$ and others do not. In the previous example, the solution $y = 1$ is asymptotically semistable, while the solution $y = -1$ is stable.

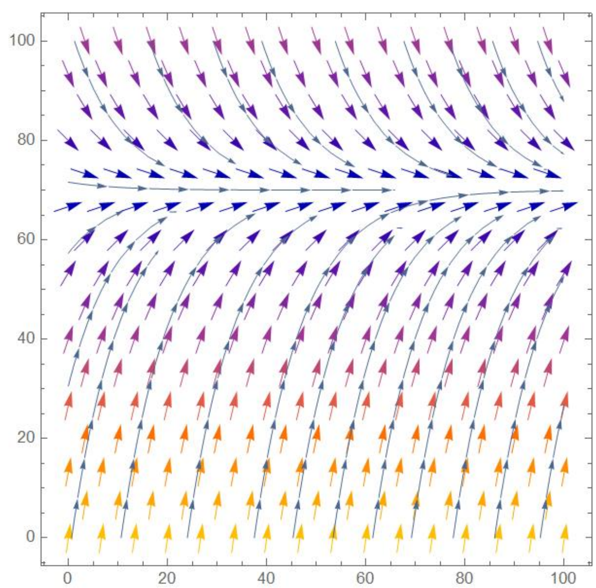
(4) One common situation is where the independent variable x or t does not appear in the equation. That is, we have an equation $y' = f(y)$. Such a differential equation is called **autonomous**. In common usage, "autonomous" means independence or having control of oneself. In this context, autonomous means independent of the variable, such as being time-independent. In physical problems, we expect any governing physical laws to be independent of time, which explains why many equations arising in practice are autonomous.

The example in item (3) is of this type. (The equation is $y' = -(y-1)^2(y+1)$.) Notice how the slope of the arrows is constant along each horizontal line. This makes sketching a direction field much easier for an autonomous equation.

(5) A typical example of an autonomous equation in physics is Newton's law of cooling. Let $T = T(t)$ denote temperature of an object (say, an apple pie or an ice cube) as a function of time. This object is placed in an ambient region of constant temperature T_a. According to Newton's law of cooling, $T(t)$ is governed by the equation

$$T' = -c(T - T_a)$$

for some constant $c > 0$. Taking $T_a = 70$ degrees Fahrenheit and $c = .1$ gives the direction field here:



You can see, both from a mathematical and physical point of view, that this equation has one equilibrium solution at $T = 70$ that is asymptotically stable.

(6) We now cover the other main topic: Euler's method. This is a simple **numerical method** for solving an initial value problem approximately. It applies to first-order equations of the form $y' = f(x, y)$, with initial value $y_0 = y(x_0)$. It is an algorithm that can easily be programmed into a computer or calculator.

Say we want to compute $y(a)$ approximately. We first pick a number of steps, say N steps. We take $\Delta x = a/N$; this is the step size. Define the following for all $n \in \{1, \ldots, N\}$:

$$x_n = x_{n-1} + \Delta x = x_0 + n\Delta x$$
$$y_n = y_{n-1} + \Delta x f(x_{n-1}, y_{n-1})$$

Then y_N is the approximate solution for $a = x_N$. See Figures 4.13 and 4.14 in the textbook to help get a better feel for the underlying concept. Notice that each pair (x_n, y_n) is computed from the preceding pair (x_{n-1}, y_{n-1}).

(7) When implementing Euler's method, it is probably easiest to organize your work as a table. Let's do an example. Take the equation $y' = y + t^2$ with initial condition $y(0) = 3$. Approximate $y(1)$ by Euler's method with $N = 5$ steps.

We organize our work into the table here:

n	x_n	y_n
0	0	3
1	.2	$3 + .2(3 + 0^2) = 3.6$
2	.4	$3.6 + .2(3.6 + .2^2) = 4.328$
3	.6	$4.328 + .2(4.328 + .4^2) = 5.2256$
4	.8	$5.2256 + .2(5.2256 + .6^2) = 6.34272$
5	1.0	$6.34272 + .2(6.34272 + .8^2) = 7.739264$

So $y(1) \approx y_5 = 7.739264$. Check your work against the table.