MAT 127 LECTURE OUTLINE WEEK 1

These lecture notes are meant to complement what is found in the textbook, to explain the same material in a slightly different way. My aim is to keep these relatively concise, while pointing you to the textbook for more details as needed.

Goal: Introduce the main topic of the first part of the course, namely **differential equations**, while reviewing some of the main ideas from Calculus A and B.

(1) Before going further, make sure to review as needed the basic concepts of **derivatives** (**differentiation**) and **integrals** (**integration**), and how to compute these for the standard functions.

Here is a very brief reference (again, these are things from Calculus A and B, and you should review the textbook as needed to feel comfortable with these concepts). The derivative of a function $f(x)$ at a point $x = a$ is the infinitesimal rate of change of f with respect to x at a, or equivalently the slope of the tangent line to f at the point a. There are two types of integrals: **indefinite integrals** and **definite integrals**. The indefinite integral of a function $f(x)$ is the set of antiderivatives of that function: all functions $F(x)$ whose derivative is equal to $f(x)$. The indefinite integral is denoted by $\int f(x)\,dx$. The definite integral of $f(x)$ from a to b is the area between the x-axis and the graph of $f(x)$ (more accurately, the *signed* area since if $f(x)$ is negative than the integral is negative too). The definite integral is denoted by $\int_a^b f(x)\,dx$.

The indefinite and definite integral seem initially to be very different concepts, but they are connected by the Fundamental Theorem of Calculus, which has two parts. The first part states that $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F(x)$ is any antiderivative of $f(x)$. In words, we can summarize this by saying that "integration undoes differentiation". The second part states that if we define $F(x) = \int_a^x f(t)\,dt$ (notice the use of the dummy variable t in the integral), then $F'(x) = f(x)$. In words, we can summarize this by saying that "differentiation undoes integration".

Make sure you can differentiate and integrate functions like $f(x) = x^3$, $f(x) = \sqrt{x}$, $f(x) = 1/\sqrt{1+x}$, $f(x) = e^x$, and so forth.

(2) A **differential equation** is an equation that may contain a function and its derivatives in addition to ordinary variables.

To explain this, let's compare a differential equation to a "normal" equation such as

$$x^2 + 2x - 3 = 0.$$

This equation contains a single variable x which is unknown; your task is to find all solutions for x. This means to find all real numbers such that, when you replace x

by that number in the equation and evaluate, you obtain equality between the left and right hand sides.

An equation can have no solutions, a single solutions, or many solutions. You may recognize the equation here as a **quadratic equation**, meaning an equation that can be put into the form $ax^2 + bx + c = 0$ for real numbers a, b, c. There are certain standard methods that can be used to solve a quadratic equation. For example, the equation here may be factored as $(x + 3)(x - 1) = 0$. By writing the equation in factored form, it becomes clear that the equation has two solutions: $x = -3$ and $x = 1$. Another method is to use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Observe that $a = 1$, $b = 2$ and $c = -3$. So the quadratic formula gives

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-3)}}{2(1)} = \frac{-2 \pm \sqrt{4 + 12}}{2} = \frac{-2 \pm 4}{2} = -3, 1.$$

In fact, any quadratic equation has either zero, one or two real solutions, which can always be found using the quadratic formula. [If you allow for solutions that are *complex* numbers, then every quadratic has one or two solutions.]

(3) In a differential equation, the unknown is a <u>function</u> rather than a <u>number</u>. This is the first main difference between a differential equation and a “normal” equation.

First, a word on notation. Most commonly, we treat x as an **independent variable** and y as a **dependent variable** that depends on x. Another way to say this is that we consider y as a function of x. This can be expressed by writing $y = f(x)$. Thus the letters x and y both can appear in a differential equation, though their roles are not interchangeable. (Sometimes we use t as the independent variable instead of x, such as when the variable represents “time”)

The other main difference between a differential equation an a “normal” equation is that we allow derivatives of the dependent variable to appear. The bulk of Calculus A was devoted to studying the **derivative** of a function $y = f(x)$, which represents the <u>instantaneous rate of change</u> of f at the time x. The derivative of function $y = f(x)$ is denoted by $f'(x)$, $\frac{dy}{dx}$, $\frac{df}{dx}(x)$, $y'(x)$, or y'. Notice that all of these notations mean the same thing, although context might make one notation most convenient in a particular circumstance. In differential equations, it is common to use the notation y' to be concise, with the understanding that the independent variable (which we take the derivative with respect to) is x.

Multiple derivatives can also appear in a differential equation, such as y'' and y'''. The **order** of a differential equation is the largest order derivative appearing in that equation.

(4) Here is our first example of a differential equation:

$$y' + xy = 0.$$

To emphasize again, x is the (independent) variable and y is a function of x: $y = f(x)$. To **solve** this differential equation means to find a function $f(x)$ such that, when we replace y by this function and simplify, we obtain equality between the left and right hand sides. Note that a differential equation can have no solutions or many solutions. Even if a solution exists, it is often very hard or even impossible to write it as a formula with the standard operations (addition, multiplication, etc.) or functions (exponential, trigonometric, etc.). Soon, we will cover certain useful methods for solving differential equations.

On the other hand, it is straightforward to verify whether a given function is or is not a solution to a differential equation. We simply substitute the function and evaluate it. This includes computing any derivatives of a function that appear in the equation, which can be done using techniques from Calculus A (the chain rule, product rule, quotient rule, etc.).

For the example above, we will propose the solution $y = e^{-x^2/2}$. The derivative y' can be computed using the chain rule along with the rules for exponential functions and polynomials:

$$y' = e^{-x^2/2}\left(\frac{-2x}{2}\right) = -xe^{-x^2/2}.$$

Substituting this into the differential equation gives

$$y' + xy = -xe^{-x^2/2} + xe^{-x^2/2} = 0.$$

So we have checked that $y = e^{-x^2/2}$ is a solution to the differential equation. This is not the only solution; you can also take $y = ae^{-x^2/2}$ for any real number a.

(5) There is one basic type of differential equation that can be solved easily. These are equations of the form $y' = g(x)$. For example, consider the equation

$$y' = e^x + 2x - 5.$$

The point is that we have "y'" on the left-hand side, and only things containing the variable x on the right-hand side.

This equation can be solved by <u>integration</u>:

$$y = \int e^x + 2x - 5\,dx = e^x + x^2 - 5x + C,$$

where C is a constant. Recall the Fundamental Theorem of Calculus: that integration and differentiation (i.e., taking a derivative) are <u>inverses</u> in a precise way. That is, each operation undoes the other one.

We have found the **general** solution to the differential equation. That is, all functions satisfying the differential equation $y' = e^x + 2x - 5$ are of the form $y = e^x + x^2 - 5x + C$. The mathematical justification for this is the fact that any two functions with the same derivative differ by a constant. By picking the value of C, we obtain **particular** solutions to the differential equation, such as $y = e^x + x^2 - 5x$, $y = e^x + x^2 - 5x - 1$ or $y = e^x + x^2 - 5x + 47/3$.

(6) Often, a differential equation comes paired with one or more **initial values** that force the equation to have a <u>unique</u> solution. Say that, in the previous example, we have the initial value $y(0) = 6$. Note that $y(0) = e^0 + 0^2 - 5(0) + C = 1 + C$. Thus $1 + C = 6$, which implies that $C = 5$. Thus this **initial-value problem** has the unique solution $y = e^x + x^2 - 5x + 5$.

(7) Let's do an example of a simple second-order differential equation, which also shows how differential equations very often arise in physics, engineering and the like. This is similar to example 4.6 in the textbook. Imagine throwing a baseball straight up in the air. Then we can measure its height/vertical position as a function of time: $y = y(t)$. Neglecting air resistance, the motion of the baseball is given by Newton's

Second Law, which states that the acceleration of the baseball is constant (with that constant being 9.8 meters per second squared). Recall that velocity is the derivative of position, and acceleration is the derivative of velocity. Thus we have

$$y'' = -9.8.$$

Since the function on the right hand side does not contain y or y', this is a differential equation that can be solved by integrating. We have

$$y' = -9.8t + C_1 \quad \text{and then} \quad y = -4.9t^2 + C_1 t + C_2$$

as the general solution. In any sort of physical problem, we expect to get a unique solution; that is, that our differential equation will allow us to accurately predict the future state of the baseball. To do this, we need an initial value problem. Here, we need two initial values (one for $y(0)$ and one for $y'(0)$), reflecting that the differential equation is a second-order equation. This should agree with our intuition that the baseball's trajectory is determined completely by its initial position and velocity.

Say we are given that the baseball has an initial height of 3 meters and an initial velocity of 14 meters/second. Then we have $14 = y'(0) = C_1$ and $3 = y(0) = C_2$. So the solution is $y = -4.9t^2 + 14t + 3$.

We can ask additional questions, such as "How long will the baseball be in the air?" and "What is the maximum height reached by the baseball?" For the first question, use the quadratic formula to get $t = \dfrac{-14 \pm \sqrt{14^2 - 4(-4.9)3}}{-9.8} \approx -.2, 3.06$. Only the positive solution $t = 3.06$ satisfies the problem physically. For the second question, observe that the maximum height is achieved when $y'(t) = -9.8t + 14 = 0$, which solves to $t = 10/7$. Then $y(10/7) = 13$ is the maximum height.

(8) As a final example, we will begin to explore the case of a differential equation

$$y' = g(x, y),$$

where g is a function of both x and y. Let's look at the equation

$$y' = y^2 - 3y + 2.$$

Notice that the derivative y' depends only on y.

When encountering a differential equation, it is a good strategy to first look for the simplest solutions or the "trivial" solutions. The simplest solutions are the constant ones. So when is the constant function $y = c$ a solution? This happens when $y'(x) = 0$. But from the differential equation, this happens when $0 = y^2 - 3y + 2$. This factors as $0 = (y - 2)(y - 1)$, so the constant solutions are $y = 1$ and $y = 2$.

Next, notice that $y'(x)$ is positive when $y > 2$ or $y < 1$, and negative when $1 < y < 2$. So this gives us a full picture of the solutions of our differential equation based on the initial value.

- If $y(0) > 2$, then $y(x)$ will increase to infinity.
- If $1 < y(0) < 2$, then $y(x)$ will decrease to 1.
- If $y(0) < 1$, then $y(x)$ increase to 1.
- If $y(0) = 1, 2$, then $y(x)$ is constant.

You might say that the constant solution $y(x) = 1$ is an "attracting" solution, while the constant solution $y(x) = 2$ is a "repelling" solution. We will explore these type of problem in more detail next week.