MAT 127

2/22/22

Last time: 2nd-order linear homogeneous differential equations w. constant coefficients

$$Ay''+By'+Cy=0,\quad y=y(x),\ A,B,C=\text{const},\ A\neq 0$$

(1) Find roots r_1, r_2 of $Ar^2+Br+C=0$

$$r_1,r_2=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

(2) $\Rightarrow y_1(x)=e^{r_1x}$, $y_2=e^{r_2x}$ are solutions and so is $y(x)=C_1e^{r_1x}+C_2e^{r_2x}$; C_1, C_2 = any const

If $r_1\neq r_2$, $\boxed{y(x)=C_1e^{r_1x}+C_2e^{r_2x}}$ is the general solution

Some Issues:

(A) What if r_1, r_2 complex? $\leftrightarrow B^2<4AC$

(B) What if $r_1=r_2$? $\leftrightarrow B^2=4AC$

(A) r_1, r_2 = complex, not real

$$r_1,r_2=\frac{-B\pm\sqrt{B^2-4AC}}{2A}\equiv \underbrace{p}_{\text{real}}\pm i\underbrace{q}_{\text{real}}$$

e^{r_1x}, e^{r_2x} are still solutions (checked last Th)

What does $e^{(p+iq)x}$ mean?

$$e^{px+iqx}=\underbrace{e^{px}}_{\text{real, OK}}\cdot e^{iqx}\ ??$$

$\boxed{\text{Euler's formula: } e^{i\theta}=\cos\theta+i\sin\theta}$

(will check in Part II of the course) $\hookrightarrow i=\sqrt{-1}$

$$\therefore\ e^{px+iqx}=e^{px}(\cos(qx)+i\sin(qx))$$

$$e^{px-iqx}=e^{px}(\cos(-qx)+i\sin(-qx))$$

$$=e^{px}(\cos(qx)-i\sin(qx))$$

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if $r_1,r_2=p\pm iq$, then

$$e^{r_1x}=e^{px}\cos(qx)+ie^{px}\sin(qx)$$

$$e^{r_2x}=e^{px}\cos(qx)-ie^{px}\sin(qx)$$

are solutions of $Ay''+By'+Cy=0$, $y=y(x)$

$\Rightarrow$ so is $y(x)=C_1e^{r_1x}+C_2e^{r_2x}$

C_1, C_2 = any constants, real or complex

~~E.g. 1: $C_1=\frac{1}{2}$, $C_2=\frac{1}{2}$~~

~~$y(x)=\frac{1}{2}e^{r_1x}+\frac{1}{2}e^{r_2x}=$~~

~~$=\frac{1}{2}e^{px}\big((\cos(qx)+i\sin(qx))+(\cos(qx)-i\sin(qx))\big)$~~

~~$y(x)=\frac{1}{2}e^{px}\cos(qx)$ is a solution~~

~~E.g. 2: $C_1=\frac{1}{2i}$, $C_2=-\frac{1}{2i}$~~

~~$y(x)=\frac{1}{2i}e^{r_1x}-\frac{1}{2i}e^{r_2x}$~~

~~$=\frac{1}{2i}e^{px}\big((\cos(qx)+i\sin(qx))-(\cos(qx)-i\sin(qx))\big)$~~

~~$=\frac{1}{2i}e^{px}\cdot 2i\sin(qx)=e^{px}\sin(qx)$~~

~~is a solution~~

Skip this; just combine complex and real.

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$y(x) = e^{px}\cos qx$, $y(x) = e^{px}\sin(qx)$
are solutions of $Ay'' + By' + Cy = 0$, $y = y(x)$

General solution:

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$
$$= \underbrace{(C_1 + C_2)}_{A_1} e^{px}\cos(qx) + \underbrace{(C_1 - C_2)i}_{A_2}\, e^{px}\sin(qx)$$

$C_1, C_2 =$ any constants $\Leftrightarrow A_1 = C_1 + C_2$, $A_2 = i(C_1 - C_2)$

$C_1 = \frac{A_1 - iA_2}{2}$, $C_2 = \frac{A_1 + iA_2}{2}$ any two constants

$\therefore$ If $r_1, r_2 = p \pm iq$ are the roots of
$Ar^2 + Br + C = 0$

then $\boxed{y(x) = C_1 e^{px}\cos(qx) + C_2 e^{px}\sin(qx)}$
is the general solution of
$Ay'' + By' + Cy = 0$, $y = y(x)$

Example 1: (a) Find the general solution of
$y'' + 2y' + 10y = 0$, $y = y(x)$

(b) Find the solution to the initial-value problem
$y'' + 2y' + 10y = 0$, $y = y(x)$, $y(\pi/2) = 1$, $y'(\frac{\pi}{2}) = 0$

2nd-order diff. eq'n
$\Rightarrow$ need 2 initial conditions

(a) $y'' + 2y' + 10y = 0$
$r^2 + 2r + 10 = 0$
$r = -1 \pm \sqrt{1 - 10} = -1 \pm 3i$
$-9 = (-1)\cdot 9$

$y(x) = C_1 e^{-x}\cos 3x + C_2 e^{-x}\sin 3x$

the general solution; 2-order $\Rightarrow$ 2 C's

(b) Find correct C_1, C_2

$$y(\tfrac{\pi}{2}) = e^{-\pi/2}\Big(C_1 \underbrace{\cos\tfrac{3\pi}{2}}_{0} + C_2 \underbrace{\sin\tfrac{3\pi}{2}}_{-1}\Big) = 1$$

$$y'(x) = C_1\big(e^{-x}(-1)\cdot\cos 3x - e^{-x}\cdot\sin 3x \cdot 3\big)$$
$$+ C_2\big(e^{-x}\cdot(-1)\cdot\sin 3x + e^{-x}\cdot\cos 3x\cdot 3\big)$$
$$= e^{-x}\big((3C_2 - C_1)\cos 3x - (3C_1 + C_2)\sin 3x\big)$$

$$y'(\tfrac{\pi}{2}) = e^{-\pi/2}\Big((3C_2 - C_1)\underbrace{\cos\tfrac{3\pi}{2}}_{0} - (3C_1 + C_2)\underbrace{\sin\tfrac{3\pi}{2}}_{-1}\Big)$$

$\therefore \begin{cases} e^{-\pi/2}\cdot C_2\cdot(-1) = 1 \rightarrow C_2 = -e^{\pi/2} \\ e^{-\pi/2}\cdot(3C_1 + C_2)\cdot(-1) = 0 \Rightarrow C_1 = -\frac{1}{3}C_2 = \frac{1}{3}e^{\pi/2} \end{cases}$

$\therefore$ $\boxed{\begin{aligned} y(x) &= \tfrac{1}{3}e^{\pi/2}e^{-x}\cos 3x - e^{\pi/2}e^{-x}\sin 3x \\ &= e^{\pi/2 - x}\big(\tfrac{1}{3}\cos 3x - \sin 3x\big) \end{aligned}}$

(either form o'k)

over $\Rightarrow$

$\therefore Ay'' + By' + Cy = 0, \quad y = y(x)$

$\hookrightarrow Ar^2 + Br + C = 0 \rightarrow$ roots r_1, r_2

$\hookrightarrow e^{r_1 x}, e^{r_2 x}, C_1 e^{r_1 x} + C_2 e^{r_2 x}$

are solutions of diff. eqn.

if $r_1 \neq r_2$, $y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

is the general solution

if $r_1, r_2 = p \pm iq$, $y(x) = C_1 e^{px}\cos(qx) + C_2 e^{px}\sin(qx)$

is the general solution

<u>Remaining case:</u> $r_1 = r_2 \Leftrightarrow B^2 = 4AC$

$e^{r_1 x} = e^{r_2 x}$ only 1 solution $\rightsquigarrow$ only 1 constant

another solution: $y(x) = x \cdot e^{r_2 x}$ (only if $r_1 = r_2$!)

Just plug in to check and use

$r_1 = -\frac{B}{2A}$ b/c $\sqrt{B^2 - 4AC} = 0$

$\Rightarrow y(x) = C_1 e^{r_1 x} + C_2 x e^{r_2 x}$

is also a solution

This is the general solution!

Example 2: Find the general solution to

$y'' + 4y' + 4y = 0, \quad y = y(x)$

Summary of all cases.