MAT 127: Calculus C, Spring 2022
Solutions to Problem Set 6 (80pts)

WebAssign Problem 1 (5pts)

Find a formula for the general term a_n of the sequence

$$-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \ldots$$

assuming the pattern continues.

First, the signs alternate beginning with -1; so the sign of a_n is $(-1)^n$. The numerator of a_n is n; the denominator is $(n+1)^2$. So $\boxed{a_n = (-1)^n \frac{n}{(n+1)^2}}$

WebAssign Problems 2,3 (8+6pts)

Determine whether each of the following sequences converges; if so, find its limit.

$$(2)\quad a_n = \frac{(-1)^n n^3}{n^3+2n^2+1}\,, \qquad (3)\quad a_n = \left(1+\frac{2}{n}\right)^n.$$

(2) Divide the top and bottom of the fraction by the highest power of n:

$$a_n = (-1)^n \frac{n^3/n^3}{n^3/n^3 + 2n^2/n^3 + 1/n^3} = (-1)^n \frac{1}{1+2/n+1/n^3}\,.$$

As $n \longrightarrow \infty$, the above fraction approaches

$$\frac{1}{1+2/n+1/n^3} \longrightarrow \frac{1}{1+2/\infty+1/\infty^3} = \frac{1}{1+0+0} = 1.$$

However, $(-1)^n$ is 1 when n is even and -1 when n is odd. So, the terms a_n with n even converge to 1, while the terms a_n with n odd converge to -1. Thus, the entire sequence a_n $\boxed{\text{does not converge}}$ anywhere (it keeps on jumping between near 1 and near -1).

(4) Replacing $n \longrightarrow \infty$ with $x \longrightarrow \infty$ makes sense in this case and

$$\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty}\left(1+\frac{1}{n/2}\right)^n = \lim_{x\longrightarrow\infty}\left(1+\frac{1}{x/2}\right)^x = \lim_{x\longrightarrow\infty}\left(\left(1+\frac{1}{x/2}\right)^{(x/2)}\right)^2$$
$$= \left(\lim_{x\longrightarrow\infty}\left(1+\frac{1}{x}\right)^x\right)^2 = e^2\,;$$

for the last equality, see p108 in Volume I.

Here is another argument. Let $b_n = \ln a_n$, so that

$$b_n = n\cdot\ln\left(1+\frac{2}{n}\right) = \frac{\ln\left(1+2\frac{1}{n}\right)}{1/n} \quad\Longrightarrow\quad \lim_{n\longrightarrow\infty} b_n = \lim_{x\longrightarrow 0}\frac{\ln(1+2x)}{x} = \lim_{x\longrightarrow 0}\frac{\frac{1}{1+2x}\cdot 2}{1} = \frac{\frac{1}{1+2\cdot 0}\cdot 2}{1} = 2.$$

The above limit computation uses l'Hospital. It is applicable here, since $\ln(1+2x), x \longrightarrow 0$ as $x \longrightarrow 0$ (the top and bottom of a fraction must *both* approach 0 or $\pm\infty$ for l'Hospital to apply). Since $b_n \longrightarrow 2$, $a_n = \mathrm{e}^{b_n} \longrightarrow \mathrm{e}^2$.

WebAssign Problem 4 (5pts)

Show that the sequence $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$ *converges and find its limit.*

Here is a quick approach that works in this case:

$$a_n = 2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} \cdot 2^{\frac{1}{8}} \cdot \ldots 2^{\frac{1}{2^n}} = 2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}} = 2^{1-\frac{1}{2^n}} = 2^1 \cdot 2^{-\frac{1}{2^n}} \longrightarrow 2 \cdot 2^{\frac{1}{\infty}} = 2 \cdot 1 = \boxed{2}$$

Here is another approach that works more generally. This sequence is recursively defined by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2a_n}$. If it converges to some number a, then

$$a = \lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} a_{n+1} = \lim_{n\longrightarrow\infty} \sqrt{2a_n} = \sqrt{2 \lim_{n\longrightarrow\infty} a_n} = \sqrt{2a};$$

so $a = \sqrt{2a}$ or $a^2 - 2a = 0$. This gives $a = 2, 0$. Since $a_n \geq 1$ for all n (square root of two numbers greater than 1 is greater than 1), the limit $a = \boxed{2}$ **provided** it exists at all.

We next show by induction that $a_n \leq 2$ for all n. This is the case for $n=1$, since $a_1 = \sqrt{2} \leq 2$. If $a_n \leq 2$, then

$$a_{n+1} = \sqrt{2a_n} \leq \sqrt{2 \cdot 2} = 2;$$

so $a_{n+1} \leq 2$ as well. This implies that $a_n \leq 2$ for all n. We next show that $a_n \leq a_{n+1}$. This is the case for $n=1$, since

$$a_1 = \sqrt{2} \leq \sqrt{2} \cdot \sqrt{\sqrt{2}} = \sqrt{2\sqrt{2}} = a_2.$$

If $a_{n-1} \leq a_n$ for some $n \geq 2$, then

$$a_n = \sqrt{2a_{n-1}} \leq \sqrt{2a_n} = a_{n+1}\,.$$

By induction, these two statements impliy that $a_n \leq a_{n+1}$ for all $n \geq 1$. So the sequence $\{a_n\}$ is increasing and bounded above. By Monotonic Sequence Theorem, $\{a_n\}$ then converges.

Note: a variation on this argument is given in the solution to Problem VI.1 on the next page.

WebAssign Problem 5 (2pts)

Determine whether the sequence $a_n = n + \dfrac{1}{n}$, $n \geq 1$ *is increasing, decreasing, or not monotonic. Is it bounded?*

As $n \longrightarrow \infty$, $1/n \longrightarrow 0$ and so $a_n \longrightarrow \infty$ and thus is $\boxed{\text{unbounded}}$ Since

$$a_{n+1} - a_n = \left(n + 1 + \frac{1}{n+1}\right) - \left(n + \frac{1}{n}\right) = 1 - \frac{1}{n(n+1)} \geq 0,$$

$\{a_n\}$ is $\boxed{\text{increasing}}$ (in fact, strictly increasing).

WebAssign Problem 6 (4pts)

Determine if the geometric series $3-4+\dfrac{16}{3}-\dfrac{64}{9}+\dots$ *converges and find its sum if converges.*
This is a geometric series with first term 3 and $r=-4/3$. Since $|r|\ge 1$, the series $\boxed{\text{diverges}}$

Problems VI.1,2 (5+5pts)

Determine whether each of the following sequences converges; if so, find its limit.

$$(1)\ a_n=\cos(2/n)\,,\qquad (2)\ a_n=\ln(2n^2+1)-\ln(n^2+1)\,.$$

(1) $\cos(2/n)\longrightarrow\cos(2/\infty)\longrightarrow\cos 0=1$; so the sequence $\boxed{\text{converges to } 1}$

(2) $$a_n=\ln\frac{2n^2+1}{n^2+1}=\ln\frac{2n^2/n^2+1/n^2}{n^2/n^2+1/n^2}=\ln\frac{2+1/n^2}{1+1/n^2}\longrightarrow\ln\frac{2+1/\infty^2}{1+1/\infty^2}=\ln\frac{2+0}{1+0}=\ln 2;$$

So the sequence $\boxed{\text{converges to } \ln 2}$

Note: If a sequence a_n is given by an explicit formula in terms of n, as in the above four problems, the first hope is to get a meaningful number just by plugging in $n\!=\!\infty$ (this number would then be the limit of the sequence). This works wonderfully in (1). However, in (2), this would give

$$\ln\infty-\ln\infty=\infty-\infty,$$

and the last expression is meaningless (it could mean anything). So, instead we first combine the logs and divide the top and bottom of the resulting fraction by the highest power of n and *only then* plug in $n\!=\!\infty$, obtaining a meaningful number. Plugging in $n\!=\!\infty$ in WA Problem 3 would give

$$\left(1+\frac{2}{\infty}\right)^{\infty}=1^{\infty}\,,$$

and the last expression is again meaningless (it could mean anything). So, we instead first replace a_n by $b_n\!=\!\ln a_n$ and $1/n$ by $x\longrightarrow 0$, apply l'Hospital, and finally plug in $x\!=\!0$, obtaining a meaningful number. Plugging in $n\!=\!\infty$ in WA Problem 2 would give $(-1)^{\infty}\infty/\infty$; this is meaningless, primarily because ∞/∞ could mean anything. In order to determine what ∞/∞ means in this case, we divide the top and bottom of the fraction by the highest power of n, finding that $|a_n|\longrightarrow 1$. As the signs of a_n alternate, this implies that the sequence a_n does not converge. However, if we had found that $|a_n|\longrightarrow 0$ (for example, if n^3 in the numerator were replaced by n^2), we would have concluded that $a_n\longrightarrow 0$, even though the terms in the sequence a_n keep on jumping between positive and negative numbers.

Problem VI.3 (15pts)

Define $\{a_n\}$ *by* $a_1=1$, $a_{n+1}=1+1/(1+a_n)$.
(a; **5pts**) *Find the first eight terms of this sequence. What do you notice about the odd terms and*

the even terms?

$$1,\ 1+\frac{1}{1+1}=\frac{3}{2}=1.5,\ 1+\frac{1}{1+\frac{3}{2}}=1+\frac{2}{5}=\frac{7}{5}=1.4,\ 1+\frac{1}{1+\frac{7}{5}}=1+\frac{5}{12}=\frac{17}{12}\approx 1.41667$$
$$1+\frac{1}{1+\frac{17}{12}}=1+\frac{12}{29}=\frac{41}{29}\approx 1.41379,\ 1+\frac{1}{1+\frac{41}{29}}=1+\frac{29}{70}=\frac{99}{70}\approx 1.41429$$
$$1+\frac{1}{1+\frac{99}{70}}=1+\frac{70}{169}=\frac{239}{169}\approx 1.41420,\ 1+\frac{1}{1+\frac{239}{169}}=1+\frac{169}{408}=\frac{577}{408}\approx 1.41422$$

The odd terms are increasing, while the even terms are decreasing and are greater than the odd terms. The odd terms are smaller than $\sqrt{2}\approx 1.41421$, while the even ones are larger. Both sets of terms appear to approach $\sqrt{2}$.

(b; **10pts**) *By considering the odd and even terms separately, show that the sequence converges and its limit is $\sqrt{2}$. This gives the continued fraction expansion*

$$\sqrt{2}=1+\frac{1}{2+\frac{1}{2+\dots}}$$

Since the odd and even terms are to be considered separately, we need to express a_{n+2} in terms of a_n:

$$\begin{aligned} a_{n+2}&=1+\frac{1}{1+a_{n+1}}=1+\frac{1}{1+1+\frac{1}{1+a_n}}=1+\frac{1}{2+\frac{1}{1+a_n}}=1+\frac{1+a_n}{3+2a_n}\\ &=\frac{4+3a_n}{3+2a_n}=\frac{3}{2}-\frac{1/2}{3+2a_n}. \end{aligned}$$

If $x>0$ and $x=(4+3x)/(3+2x)$, then $2x^2-4=0$, so that $x=\sqrt{2}$. Thus,

$$f(x)=\frac{4+3x}{3+2x}=\frac{3}{2}-\frac{1/2}{3+2x}$$

is a strictly increasing function of $x>0$ such that $f(\sqrt{2})=\sqrt{2}$ (so $f(x)<\sqrt{2}$ if and only if $x<\sqrt{2}$). Furthermore,

$$f(x)-x=\frac{4+3x}{3+2x}-x=\frac{4-2x^2}{3+2x}\quad\begin{cases}>0, & \text{if } x<\sqrt{2};\\ <0, & \text{if } x>\sqrt{2}.\end{cases}$$

So $x<f(x)<\sqrt{2}$ if $x<\sqrt{2}$ and $x>f(x)>\sqrt{2}$ if $x>\sqrt{2}$. Since $a_{n+2}=f(a_n)$ and $a_1<\sqrt{2}$, $a_n<a_{n+2}<\sqrt{2}$ for all n odd. Since $a_2>\sqrt{2}$, $a_n>a_{n+2}>\sqrt{2}$ for all n even. So, the sequence $\{a_{2n-1}\}_{n\ge 1}$ is increasing and bounded above and thus converges by the Monotonic Sequence Theorem. The sequence $\{a_{2n}\}_{n\ge 1}$ is decreasing and bounded below and thus also converges by the Monotonic Sequence Theorem. Each of the sequences must converge to a non-negative number a such that $a=f(a)$; the only such number is $a=\sqrt{2}$. Since the even and odd terms converge to $\sqrt{2}$, the entire sequence converges to $\sqrt{2}$.

Note: until it is established that the entire sequence converges, it is wrong to assume that the limit is a number a such that $a=1+1/(1+a)$. If $a_{n+1}=f(a_n)$, it could be the case that the odd and even terms converge to different numbers a_o and a_e such that $a_o=f(a_e)$ and $a_e=f(a_0)$. This cannot happen in this case because f is a strictly increasing function for $x>0$.

Problem VI.4 (5pts)

Find the sum of the series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9}+\frac{1}{12}+\dots.$$

where the terms are the reciprocals of the positive integers whose only prime factors are 2s and 3s.

This sum is

$$\begin{aligned}\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots\right)\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\dots\right) &= \left(\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\right)\left(\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n\right)\\ &= \left(\frac{1}{1-\frac{1}{2}}\right)\left(\frac{1}{1-\frac{1}{3}}\right) = \frac{2}{2-1}\cdot\frac{3}{3-1} = \boxed{3}\end{aligned}$$

Problem G (10pts)

Show (by induction) that the n-th Fibonacci number f_n defined at the top of p445 is given by

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right). \tag{1}$$

Is this consistent with the textbook's answer to (3) on p445 and why?

By definition,

$$f_n = f_{n-1}+f_{n-2} \ \text{ if } n\ge 2, \qquad f_0=0,\ f_1=1. \tag{2}$$

We need check that RHS of (1) satisfies the two initial conditions and the recursion in (2):

$$\begin{aligned}
f_0 &\overset{?}{=} \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0\right) = \frac{1}{\sqrt{5}}(1-1) = 0; \ \checkmark\\
f_1 &\overset{?}{=} \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right) = \frac{1}{\sqrt{5}}\cdot\frac{2\sqrt{5}}{2} = 1; \ \checkmark\\
f_{n-1}+f_{n-2} &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(\frac{1+\sqrt{5}}{2}+1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\left(\frac{1-\sqrt{5}}{2}+1\right)\right)\\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\left(\frac{3-\sqrt{5}}{2}\right)\right)\\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\left(\frac{1-\sqrt{5}}{2}\right)^2\right)\\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\right) = f_n. \ \checkmark
\end{aligned}$$

In the last equation, we assume that $n \geq 2$. By (1),

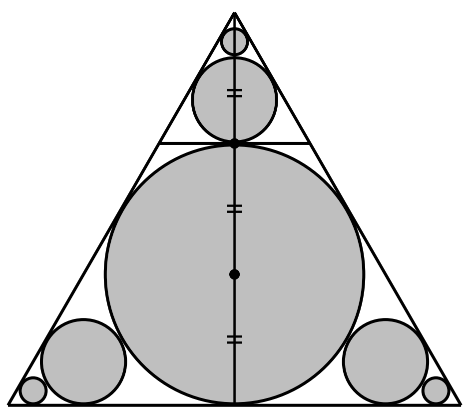
$$\lim_{n\longrightarrow\infty} \frac{f_{n+1}}{f_n} = \lim_{n\longrightarrow\infty} \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{-1+\sqrt{5}}{2}\right)^{n+1}}{\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{-1+\sqrt{5}}{2}\right)^{n}} = \lim_{n\longrightarrow\infty} \frac{\frac{1+\sqrt{5}}{2} - \frac{-1+\sqrt{5}}{2}\left(\frac{-1+\sqrt{5}}{1+\sqrt{5}}\right)^{n}}{1 - \left(\frac{-1+\sqrt{5}}{1+\sqrt{5}}\right)^{n}}$$

$$= \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} \cdot 0}{1-0} = \frac{1+\sqrt{5}}{2}\,.$$

This is the textbook answer to (3) on p445, so the formula (1) is consistent with it.

Problem H (10pts)

In the figure below, there are infinitely many inscribed circles in an equilateral triangle, each touching other circles and sides of the triangle. Find the total area enclosed by the circles if the sides of the triangle are of length 1.



The length of any of the altitudes in this triangle is $\sin(\pi/3) = \sqrt{3}/2$. Since the medians in any triangle intersect at a point that divides each median in the ration of 2 : 1 from the vertex, the radius of the largest circle in the above triangle is

$$r_0 = \frac{1}{3} \cdot \frac{\sqrt{3}}{2} = \frac{1}{2\sqrt{3}}\,.$$

Thus, the top sub-triangle is 1/3 the size of the large triangle (length-wise), so the second-largest radius is $r_1 = (1/3)r_0$. The radius of each successive circle must then be 1/3 the radius of the preceding circle:

$$r_n = \frac{1}{3} r_{n-1} = \frac{1}{3^n} r_0 = \frac{1}{3^n} \cdot \frac{1}{2\sqrt{3}}\,.$$

So the total area enclosed by the circles in each of the three chains, not including the largest circle, is

$$\sum_{n=1}^{\infty} \pi r_n^2 = \pi \sum_{n=1}^{\infty} \frac{1}{3^{2n}} \cdot \frac{1}{4\cdot 3} = \pi \frac{\frac{1}{9}\cdot\frac{1}{12}}{1-\frac{1}{9}} = \frac{\pi}{96}$$

Since there are three such chains and one central circle, the total area enclosed by the circles is

$$\pi \cdot \frac{1}{4\cdot 3} + 3\frac{\pi}{96} = \frac{\pi}{12} + \frac{3\pi}{96} = \boxed{\frac{11}{96}\pi}$$