MAT 127: Calculus C, Spring 2020
Solutions to Problem Set 4 (100pts)

WebAssign Problem 1 (8pts)

A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420.

(a) *Find an expression for the number of bacteria after t hours.*
Let $y(t)$ be the number of cells at time t, measured in hours. We assume that $y(t)$ grows exponentially:

$$y(t) = y(0)\mathrm{e}^{rt} = 100\mathrm{e}^{rt}\,,$$

where r is the relative growth rate; it is a constant. Since $y(1) = 420$,

$$420 = 100\mathrm{e}^{r} \qquad \Longleftrightarrow \qquad 4.2 = \mathrm{e}^{r} \qquad \Longleftrightarrow \qquad r = \ln 4.2.$$

So the number of cells after t hours is

$$y(t) = 100\mathrm{e}^{(\ln 4.2)t} = 100(\mathrm{e}^{\ln 4.2})^{t} = \boxed{100 \cdot 4.2^{t}}$$

(b) *Find the number of bacteria after 3 hours.*
Plug in $t = 3$ into the above formula:

$$y(3) = 100 \cdot 4.2^{3} = 7,408.8\,.$$

Since the number of cells is integer, their number after 3 hours is $\boxed{7{,}409}$

(c) *Find the rate of growth after 3 hours.*
The rate of growth at time t is $ry(t) = (\ln 4.2)y(t)$; so after 3 hours the rate of growth is

$$(\ln 4.2) \cdot 7,408.8 \approx \boxed{10{,}632 \text{ cells per hour}}$$

(d) *When will the population reach 10,000?*
We need to find t such that

$$y(t) = 100 \cdot 4.2^{t} = 10,000 \qquad \Longleftrightarrow \qquad 4.2^{t} = 100 \qquad \Longleftrightarrow \qquad \boxed{t = \log_{4.2} 100 = \tfrac{\ln 100}{\ln 4.2} \approx 3.21 \text{ hrs}}$$

WebAssign Problem 2 (9pts)

The half-life of cesium-137 is 30 years. Suppose we have a 100 mg sample.
(a) *Find the mass that remains after t years.*
(b) *How much of the sample remains after 100 years?*
(c) *After how long will only 1 mg remain?*

Let $y(t)$ be the portion of cesium-137 remaining after t years. Since we assume exponential decay,

$$y(t) = y(0)\mathrm{e}^{rt} = \mathrm{e}^{rt}\,,$$

where r is the relative decay rate; it is a constant. Since $y(30) = .5$,

$$e^{30r} = \frac{1}{2} \qquad \Longleftrightarrow \qquad 30r = \ln 2^{-1} = -\ln 2 \qquad \Longleftrightarrow \qquad r = -(\ln 2)/30.$$

Thus,
$$y(t) = e^{-(\ln 2)t/30} = \left(e^{\ln 2}\right)^{(-t/30)} = 2^{-t/30}.$$

Since we start with 100 mg, the amount left after t years is $\boxed{100 \cdot 2^{-t/30} \text{ mg}}$. In particular, the amount remaining after 100 years is

$$100 \cdot 2^{-100/30} = \boxed{100 \cdot 2^{-10/3} \approx 9.92 \text{ mg}}$$

The number of years t it will take for the sample to decline to 1 mg is given by

$$y(t) = 100 \cdot 2^{-t/30} = 1 \quad \Longleftrightarrow \quad 2^{-t/30} = 1/100 \quad \Longleftrightarrow \quad -t/30 = \log_2 10^{-2} = -2\log_2 10$$
$$\Longleftrightarrow \quad \boxed{t = 60\log_2 10 = 60\frac{\ln 10}{\ln 2} \approx 199.32 \text{ years}}$$

WebAssign Problem 3 (4pts)

How long will it take an investment to double in value if the interest rate is 6% compounded continuously?

Let $y(t)$ be the ratio of the balance after t years and of the original investment. Since $y(t)$ grows exponentially,
$$y(t) = y(0)e^{rt} = e^{rt},$$
where r is the relative growth rate. In this case, $r = 6\% = .06$, so $y(t) = e^{.06t}$. We need to find t so that $y(t) = 2$, i.e.

$$2 = e^{.06t} \qquad \Longleftrightarrow \qquad \ln 2 = .06t \qquad \Longleftrightarrow \qquad \boxed{t = \tfrac{\ln 2}{.06} \approx 11.55 \text{ years}}$$

WebAssign Problem 4 (10pts)

A lake with estimated carrying capacity of 10,000 fish is stocked with 400 fish. The number of fish tripled in the first year.
(a) *Assuming the size of the population satisfies the logistic equation, find an expression for the size of the population after t years.*
(b) *How long will it take for the fish population to increase 5,000?*

(a) Let $P(t)$ be the number of fish after t years. Since $P(t)$ is assumed to satisfy the logistic equation
$$P(t) = \frac{K}{1 + \frac{K - P(0)}{P(0)}e^{-rt}} = \frac{10,000}{1 + \frac{10,000-400}{400}e^{-rt}} = \frac{10,000}{1 + 24e^{-rt}};$$

see equation (4) on p534 (you should also be able to derive this). Since $y(1) = 3 \cdot y(0)$,

$$1,200 = \frac{10,000}{1+24e^{-rt}} \iff 1+24e^{-r} = \frac{10,000}{1,200} = \frac{25}{3} \iff e^{-r} = \frac{1}{24}\cdot\frac{22}{3} = \frac{11}{36}$$
$$\iff -r = \ln\left(\frac{11}{36}\right).$$

Thus, the size of the population after t years is given by

$$y(t) = \frac{10,000}{1+24e^{-\ln(36/11)t}} = \frac{10,000}{1+24e^{\ln(11/36)t}} = \boxed{\frac{10,000}{1+24(11/36)^t}}$$

(b) We need to find t so that $y(t) = 5,000$, i.e.

$$5,000 = \frac{10,000}{1+24(11/36)^t} \iff 1+24(11/36)^t = \frac{10,000}{5,000} = 2 \iff (11/36)^t = \frac{1}{24}$$
$$\iff t = \log_{11/36}(1/24) = \frac{\ln(1/24)}{\ln(11/36)} = \frac{-\ln(24)}{-\ln(36/11)} = \boxed{\frac{\ln(24)}{\ln(36/11)} \approx 2.68 \text{ years}}$$

Remark: On the exams, you will need to leave your answers in an exact form, **as simple as possible**, even if they involve exponentials and logs.

WebAssign Problem 5 (4pts)

Find the general solution to the differential equation

$$y'' - 13y' + 42y = 0, \qquad y = y(x).$$

The associated quadratic equation is

$$r^2 - 13r + 42 = 0.$$

This gives $(r-6)(r-7) = 0$, so the roots are $r = 6, 7$ and the general solution of the differential equation is $\boxed{y(x) = C_1 e^{6x} + C_2 e^{7x}}$

WebAssign Problem 6 (4pts)

Find the general solution to the differential equation

$$2y'' + 3y' = 0, \qquad y = y(x).$$

This is the same as $2y'' + 3y' + 0y = 0$, so the associated quadratic equation is

$$2r^2 + 3r + 0 = 0.$$

This gives $r(2r+3)=0$, so the roots are $r=0,-3/2$ and the general solution of the differential equation is

$$y(x)=C_1\mathrm{e}^{0x}+C_2\mathrm{e}^{-(3/2)x}=\boxed{C_1+C_2\mathrm{e}^{-3x/2}}$$

Note: This equation can be solved in a different way. Letting $z=y'$, the original equation becomes

$$\begin{aligned}
2z'+3z=0,\quad z=z(x)\quad &\Longleftrightarrow\quad 2\frac{\mathrm{d}z}{\mathrm{d}x}=-3z\quad\Longleftrightarrow\quad\frac{\mathrm{d}z}{z}=-\frac{3}{2}\mathrm{d}x\quad\Longleftrightarrow\quad\int\frac{\mathrm{d}z}{z}=-\int\frac{3}{2}\mathrm{d}x\\
&\Longleftrightarrow\quad\ln|z|=-\frac{3}{2}x+C\quad\Longleftrightarrow\quad|z|=\mathrm{e}^{-3x/2+C}=A\mathrm{e}^{-3x/2}\\
&\Longleftrightarrow\quad y'=z=\pm A\mathrm{e}^{-3x/2}=C\mathrm{e}^{-3x/2}.
\end{aligned}$$

Integrating the last expression, we obtain

$$y(x)=C\cdot\frac{1}{-3/2}\mathrm{e}^{-3x/2}+C_2=C_1\mathrm{e}^{-3x/2}+C_2.$$

This approach works only because the equation involves y'' and y', and not y; so it is really an equation for y'. While this approach is longer, it does not require remembering how to solve second-order linear homogeneous differential equations with constant coefficients (though you should certainly remember this).

WebAssign Problem 7 (4pts)

Find the general solution to the differential equation

$$y''+2y'+y=0,\quad y=y(x).$$

The associated quadratic equation is

$$r^2+2r+1=0.$$

This gives $(r+1)^2=0$, so the roots are $r_1=r_2=-1$ and the general solution of the differential equation is

$$y(x)=C_1\mathrm{e}^{-x}+C_2x\mathrm{e}^{-x}=\mathrm{e}^{-x}\big(C_1+C_2x\big).$$

WebAssign Problem 8 (7pts)

Find the solution to the initial-value problem

$$y''-2y'+5y=0,\quad y=y(x),\qquad y(\pi/2)=0,\quad y'(\pi/2)=2.$$

The associated quadratic equation is

$$r^2-2r+5=0.$$

So the roots are $r_1, r_2 = 1 \pm 2\mathrm{i}$ and the real form of the general solution of the differential equation is

$$y(x) = C_1 \mathrm{e}^x \cos 2x + C_2 \mathrm{e}^x \sin 2x = \mathrm{e}^x \big(C_1 \cos 2x + C_2 \sin 2x \big).$$

We need to find C_1, C_2 so that $y(\pi/2) = 0$, $y'(\pi/2) = 2$. For this, first compute $y'(x)$:

$$\begin{aligned} y'(x) &= C_1 \big(\mathrm{e}^x \cos 2x - 2\mathrm{e}^x \sin 2x \big) + C_2 \big(\mathrm{e}^x \sin 2x + 2\mathrm{e}^x \cos 2x \big) \\ &= \mathrm{e}^x \big((C_1 + 2C_2) \cos 2x + (C_2 - 2C_1) \sin 2x \big). \end{aligned}$$

Now plug in $x = \pi/2$ into $y(x)$ and $y'(x)$ and use $\cos(\pi) = -1$, $\sin(\pi) = 0$, and the initial conditions:

$$\begin{cases} \mathrm{e}^{\pi/2}(-C_1 + C_2 \cdot 0) = 0 \\ \mathrm{e}^{\pi/2}(-(C_1 + 2C_2) + (C_2 - 2C_1) \cdot 0) = 2 \end{cases} \iff \begin{cases} C_1 = 0 \\ C_1 + 2C_2 = -2\mathrm{e}^{-\pi/2} \end{cases}$$

$$\iff C_1 = 0,\ C_2 = -\mathrm{e}^{-\pi/2}.$$

So the solution to the initial-value problem is

$$\boxed{y(x) = -\mathrm{e}^{-\pi/2} \mathrm{e}^x \sin 2x = -\mathrm{e}^{x - \pi/2} \sin 2x}$$

Note: you can double-check this by computing y', y'' and plugging these into the original equation and also by checking the initial conditions.

Problem D (10pts)

(a) *Use Euler's Formula to obtain the cosine/sine addition/subtraction formulas:*

$$\cos(\alpha \pm \beta) = \cos\alpha \cdot \cos\beta \mp \sin\alpha \cdot \sin\beta \qquad \text{and} \qquad \sin(\alpha \pm \beta) = \sin\alpha \cdot \cos\beta \pm \cos\alpha \cdot \sin\beta.$$

(b) *Use the last two formulas to obtain the cosine/sine double angle formulas:*

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \qquad \text{and} \qquad \sin 2\theta = 2\cos\theta \cdot \sin\theta.$$

(a; **6pt**) Euler's Formula says

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\,\sin\theta, \qquad \cos\theta = \Re \mathrm{e}^{\mathrm{i}\theta}, \qquad \sin\theta = \Im \mathrm{e}^{\mathrm{i}\theta}.$$

Combining this with exponential rules, we obtain

$$\begin{aligned} \cos(\alpha \pm \beta) + \mathrm{i} \sin(\alpha \pm \beta) &= \mathrm{e}^{\mathrm{i}(\alpha \pm \beta)} = \mathrm{e}^{\mathrm{i}\alpha \pm \mathrm{i}\beta} = \mathrm{e}^{\mathrm{i}\alpha} \cdot \mathrm{e}^{\pm \mathrm{i}\beta} = \big(\cos\alpha + \mathrm{i} \sin\alpha \big) \cdot \big(\cos\beta \pm \mathrm{i} \sin\beta \big) \\ &= \big(\cos\alpha \cdot \cos\beta \mp \sin\alpha \cdot \sin\beta \big) + \mathrm{i} \big(\sin\alpha \cdot \cos\beta \pm \cos\alpha \cdot \sin\beta \big). \end{aligned}$$

Taking the real and imaginary parts of the LHS and RHS of the above equation, we obtain the two identities in (a).

(b; **4pt**) Taking $\alpha = \beta = \theta$ in the $+$ case of the first identity in (a) and using $\cos^2\theta + \sin^2\theta = 1$, we obtain

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta - \big(1 - \cos^2\theta \big) = 2\cos^2\theta - 1 = 2\big(1 - \sin^2\theta \big) - 1 = 1 - 2\sin^2\theta.$$

This gives the first three equalities in (b). Taking $\alpha = \beta = \theta$ in the $+$ case of the second identity in (a), we obtain

$$\sin(2\theta) = \sin\theta \cdot \cos\theta + \cos\theta \cdot \sin\theta = 2\cos\theta \cdot \sin\theta.$$

This gives the last equality in (b).

Problem E (40pts)

By Problem B on HW2, the first-order differential equation

$$y' - by = f(x), \quad y = y(x), \qquad b = const,$$

can be solved by multiplying both sides by e^{-bx}. *This equation then becomes*

$$\big(e^{-bx}y\big)' = e^{-bx}f(x)$$

and can be solved by integrating both sides. Note that b *is the root of the associated linear equation* $r-b=0$. *This approach has an analogue for second-order inhomogeneous linear equations*

$$y'' + by' + cy = f(x), \quad y = y(x), \qquad b, c = const. \tag{1}$$

(a; **5pts**) *If* r_1, r_2 *are the two roots of the quadratic equation* $r^2+br+c=0$ *associated to (1), show that*

$$\big(e^{(r_1-r_2)x}(e^{-r_1x}y)'\big)' = e^{-r_2x}(y''+by'+cy). \tag{2}$$

This is the product rule applied twice:

$$\begin{aligned}
(e^{-r_1x}y)' &= e^{-r_1x}y' + (e^{-r_1x})'y = e^{-r_1x}(y'-r_1y);\\
\big(e^{(r_1-r_2)x}(e^{-r_1x}y)'\big)' &= \big(e^{-r_2x}(y'-r_1y)\big)' = e^{-r_2x}(y'-r_1y)' + (e^{-r_2x})'(y'-r_1y)\\
&= e^{-r_2x}\big(y''-r_1y'-r_2(y'-r_1y)\big) = e^{-r_2x}\big(y''-(r_1+r_2)y'+r_1r_2y\big)\\
&= e^{-r_2x}(y''+by'+cy)\,,
\end{aligned}$$

since $r_1+r_2=-b$ and $r_1r_2=c$.

By (2), equation (1) is equivalent to

$$\big(e^{(r_1-r_2)x}(e^{-r_1x}y)'\big)' = e^{-r_2x}f(x), \quad y = y(x), \tag{3}$$

which can be solved by integrating twice.

(b; **15pts**) *Find the general solution* $y=y(x)$ *of the differential equation*

$$y''+5y'+4y = e^{-x}. \tag{4}$$

In this case, the associated quadratic polynomial is

$$r^2+5r+4 = (r+1)(r+4).$$

Thus, the two roots are $r_1=-1$ and $r_2=-4$, and

$$\big(e^{((-1)-(-4))x}(e^{-(-1)x}y)'\big)' = e^{-(-4)x}(y''+5y'+4y). \tag{5}$$

Multiplying both sides of (4) by $e^{4x}=e^{-(-4)x}$ and using (5), we obtain

$$y''+5y'+4y = e^{-x} \iff e^{4x}(y''+5y'+4y) = e^{3x} \iff \big(e^{3x}(e^{x}y)'\big)' = e^{3x}.$$

Integrating twice, we obtain

$$\mathrm{e}^{3x}(e^x y)' = \int \mathrm{e}^{3x}\mathrm{d}x = \frac{1}{3}\mathrm{e}^{3x} + C_1 \quad\Longleftrightarrow\quad (\mathrm{e}^x y)' = \frac{1}{3} + C_1\mathrm{e}^{-3x}$$
$$\Longleftrightarrow\quad \mathrm{e}^x y(x) = \int \left(\frac{1}{3} + C_1\mathrm{e}^{-3x}\right)\mathrm{d}x = \frac{1}{3}x - \frac{1}{3}C_1\mathrm{e}^{-3x} + C_2.$$

Since we can replace $(-C_1/3)$ with C_1, the general solution of (4) is

$$\boxed{y(x) = \tfrac{1}{3}x\mathrm{e}^{-x} + C_1\mathrm{e}^{-4x} + C_2\mathrm{e}^{-x}}$$

(c; **20pts**) *Find the general real solution $y = y(x)$ of the differential equation*

$$y'' + 4y = 4\cos 2x. \tag{6}$$

Here is one approach. By Euler's formula, the general real solution $y = y(x)$ of this equation is given by $y = \mathrm{Re}\, z$, where $z = z(x)$ is the complex general solution of

$$z'' + 4z = 4\mathrm{e}^{2\mathrm{i}x}. \tag{7}$$

The associated polynomial for this equation is

$$r^2 + 0 \cdot r + 4 = (r - 2\mathrm{i})(r + 2\mathrm{i}).$$

Thus, the two roots are $r_1 = 2\mathrm{i}$ and $r_2 = -2\mathrm{i}$, and

$$\big(\mathrm{e}^{((2\mathrm{i})-(-2\mathrm{i}))x}(\mathrm{e}^{-(2\mathrm{i})x}z)'\big)' = \mathrm{e}^{-(-2\mathrm{i})x}(z'' + 4z). \tag{8}$$

Multiplying both sides of (7) by $\mathrm{e}^{2\mathrm{i}x} = \mathrm{e}^{-(-2\mathrm{i})x}$ and using (8), we obtain

$$z'' + 4z = 4\mathrm{e}^{2\mathrm{i}x} \quad\Longleftrightarrow\quad \mathrm{e}^{2\mathrm{i}x}(z'' + 4z) = 4\mathrm{e}^{4\mathrm{i}x} \quad\Longleftrightarrow\quad \big(\mathrm{e}^{4\mathrm{i}x}(\mathrm{e}^{-2\mathrm{i}x}z)'\big)' = 4\mathrm{e}^{4\mathrm{i}x}.$$

Integrating twice, we obtain

$$e^{4\mathrm{i}x}(e^{-2\mathrm{i}x}z)' = \int 4\mathrm{e}^{4\mathrm{i}x}\mathrm{d}x = \frac{4}{4\mathrm{i}}\mathrm{e}^{4\mathrm{i}x} + C_1 = -\mathrm{i}\mathrm{e}^{4\mathrm{i}x} + C_1 \quad\Longleftrightarrow\quad (\mathrm{e}^{-2\mathrm{i}x}z)' = -\mathrm{i} + C_1\mathrm{e}^{-4\mathrm{i}x}$$
$$\Longleftrightarrow\quad \mathrm{e}^{-2\mathrm{i}x}z = \int \big(-\mathrm{i} + C_1\mathrm{e}^{-4\mathrm{i}x}\big)\mathrm{d}x = -\mathrm{i}x + \frac{C_1}{-4\mathrm{i}}\mathrm{e}^{-4\mathrm{i}x} + C_2.$$

Since we can replace $-C_1/4\mathrm{i}$ with C_1, the general solution of (7) is

$$z(x) = -\mathrm{i}x\mathrm{e}^{2\mathrm{i}x} + C_1\mathrm{e}^{-2\mathrm{i}x} + C_2\mathrm{e}^{2\mathrm{i}x}.$$

Taking the real part of this equation and modifying the constants, we obtain

$$\boxed{y(x) = \mathrm{Re}\, z(x) = x\sin 2x + C_1\cos 2x + C_2\sin 2x}$$

Here is another approach. The associated polynomial and roots for the original equation are the same as for its complex version. Thus, (8) holds with z replaced by y, and

$$y''+4y=4\cos 2x \quad\Longleftrightarrow\quad \mathrm{e}^{2\mathrm{i}x}(y''+4y)=4\mathrm{e}^{2\mathrm{i}x}\cos 2x \quad\Longleftrightarrow\quad \big(\mathrm{e}^{4\mathrm{i}x}(\mathrm{e}^{-2\mathrm{i}x}y)'\big)'=4\mathrm{e}^{2\mathrm{i}x}\cos 2x.$$

Integrating the last expression once, we obtain

$$\begin{aligned}\mathrm{e}^{4\mathrm{i}x}(\mathrm{e}^{-2\mathrm{i}x}y)' &= \int 4\mathrm{e}^{2\mathrm{i}x}\cos 2x\,\mathrm{d}x = 4\int\cos^2 2x\,\mathrm{d}x+4\mathrm{i}\int\cos 2x\,\sin 2x\,\mathrm{d}x\\ &= 2\int(\cos 4x+1)\mathrm{d}x+2\mathrm{i}\int\sin 4x\,\mathrm{d}x=\frac{1}{2}\sin 4x+2x-\frac{\mathrm{i}}{2}\cos 4x+C_1=-\frac{\mathrm{i}}{2}\mathrm{e}^{4\mathrm{i}x}+2x+C_1.\end{aligned}$$

The second and last equalities above follow from Euler's formula, applied in opposite directions; the third equality uses the half-angle trigonometric formulas. This gives

$$(\mathrm{e}^{-2\mathrm{i}x}y)'=-\frac{\mathrm{i}}{2}+2x\mathrm{e}^{-4\mathrm{i}x}+C_1\mathrm{e}^{-4\mathrm{i}x}\,.$$

Finally, using integration by parts, we obtain

$$\begin{aligned}\mathrm{e}^{-2\mathrm{i}x}y &= \int\Big(-\frac{\mathrm{i}}{2}+2x\mathrm{e}^{-4\mathrm{i}x}+C_1\mathrm{e}^{-4\mathrm{i}x}\Big)\mathrm{d}x=-\frac{\mathrm{i}}{2}x-\frac{C_1}{4\mathrm{i}}\mathrm{e}^{-4\mathrm{i}x}-\frac{1}{2\mathrm{i}}\int x\,\mathrm{d}\mathrm{e}^{-4\mathrm{i}x}\\ &= -\frac{\mathrm{i}}{2}x-\frac{C_1}{4\mathrm{i}}\mathrm{e}^{-4\mathrm{i}x}+\frac{\mathrm{i}}{2}\Big(x\mathrm{e}^{-4\mathrm{i}x}-\int\mathrm{e}^{-4\mathrm{i}x}\mathrm{d}x\Big)=-\frac{\mathrm{i}}{2}x-\frac{C_1}{4\mathrm{i}}\mathrm{e}^{-4\mathrm{i}x}+\frac{\mathrm{i}}{2}\Big(x\mathrm{e}^{-4\mathrm{i}x}+\frac{1}{4\mathrm{i}}\mathrm{e}^{-4\mathrm{i}x}+C_2\Big)\\ &\Longrightarrow\quad y(x)=-\frac{\mathrm{i}}{2}x\big(\mathrm{e}^{2\mathrm{i}x}-\mathrm{e}^{-2\mathrm{i}x}\big)+\frac{1-C_1}{4\mathrm{i}}\mathrm{e}^{-2\mathrm{i}x}+\frac{\mathrm{i}C_2}{2}\mathrm{e}^{2\mathrm{i}x}\,.\end{aligned}$$

Replacing the constant $(1-C_1)/4\mathrm{i}$ with C_1 and $\mathrm{i}C_2/2$ with C_2 gives

$$y(x)=-\frac{\mathrm{i}}{2}x\big(\mathrm{e}^{2\mathrm{i}x}-\mathrm{e}^{-2\mathrm{i}x}\big)+C_1\mathrm{e}^{-2\mathrm{i}x}+C_2\mathrm{e}^{2\mathrm{i}x}=x\sin 2x+C_1\mathrm{e}^{-2\mathrm{i}x}+C_2\mathrm{e}^{2\mathrm{i}x}\,.$$

As before, the complex form $C_1\mathrm{e}^{-2\mathrm{i}x}+C_2\mathrm{e}^{2\mathrm{i}x}$ is equivalent to the real form $A_1\cos 2x+A_2\sin 2x$.

Remarks: (1) When the inhomogeneous term, i.e. RHS in (6), is $\cos\omega t$ or $\sin\omega t$, the first approach, i.e. complexifying the differential equation, is generally faster, but riskier if you are not used to complex numbers. Note that if the inhomogeneous term is $\sin\omega t$, you would need to take the imaginary part of the complex solution.
(2) The complex form $C_1\mathrm{e}^{px+\mathrm{i}qx}+C_2\mathrm{e}^{px-\mathrm{i}qx}$ of the general solution of a differential equation is always equivalent to the real form $A_1\mathrm{e}^{px}\cos qx+A_2\mathrm{e}^{px}\sin qx$.