MAT 127: Calculus C, Spring 2022
Solutions to Problem Set 11 (110pts)

WebAssign Problems 1-3 (10+5+4pts)

Find the Taylor series expansion for each of the following functions around the given value of $x=a$ and determine the radius and interval of convergence.

$$(1)\quad f(x)=1/x,\quad a=-3,\qquad (2)\quad f(x)=\mathrm{e}^x+2\mathrm{e}^{-x},\quad a=0\qquad (3)\quad \int\frac{\mathrm{e}^x-1}{x}\mathrm{d}x\,,\quad a=0.$$

(1) In this case, we can compute all derivatives. By induction,

$$f^{\langle n\rangle}(x)=\frac{(-1)^n n!}{x^{n+1}}\,.$$

This is true for $n\!=\!0$, since $f^{\langle 0\rangle}(x)\!=\!f(x)\!=\!1/x=(-1)^0 0!/x^{0+1}$. If this holds for some n, then

$$f^{\langle n+1\rangle}(x)=\big(f^{\langle n\rangle}(x)\big)'=\left(\frac{(-1)^n n!}{x^{n+1}}\right)'=-(n\!+\!1)\frac{(-1)^n n!}{x^{n+2}}=\frac{(-1)^{n+1}(n\!+\!1)!}{x^{(n+1)+1}}\,,$$

since

$$-(-1)^n=(-1)^{n+1}\qquad\text{and}\qquad (n\!+\!1)\cdot n!=(n\!+\!1)!$$

So, we have checked the above formula for $f^{\langle n\rangle}$ for the base case $n\!=\!0$ and that if it holds for some n, then it holds for $n\!+\!1$. Thus, it holds for all n and

$$f^{\langle n\rangle}(-3)=\frac{(-1)^n n!}{(-3)^{n+1}}=-\frac{n!}{3^{n+1}}.$$

Thus, by the main Taylor series formula

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{\langle n\rangle}(-3)}{n!}\big(x-(-3)\big)^n=\boxed{\sum_{n=0}^{\infty}\frac{-1}{3^{n+1}}(x+3)^n}$$

This is a geometric series with the ratio $r=(x+3)/3$ and so converges whenever $|x+3|/3<1$. So, the radius of convergence is $\boxed{3}$ and the interval of convergence is $\boxed{(-6,0)}$

Note: Since the power series in this case is a geometric series, we can sum it up using the geometric-series formula:

$$\sum_{n=0}^{\infty}\frac{-1}{3^{n+1}}(x+3)^n=\frac{\text{initial}}{1-r}=\frac{-1/3}{1-(x+3)/3}=\frac{-1}{3-(x+3)}=\frac{-1}{-x}=\frac{1}{x}\,.$$

This confirms that the above Taylor series expansion for $f(x)=1/x$ is correct.

(2) Since $\mathrm{e}^x=\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$ for all x,

$$\mathrm{e}^x+2\mathrm{e}^{-x}=\sum_{n=0}^{\infty}\frac{x^n}{n!}+2\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}=\boxed{\sum_{n=0}^{\infty}\frac{1+2(-1)^n}{n!}x^n}$$

for all x. So the interval of convergence is $\boxed{(-\infty, \infty)}$ and the radius of convergence is $\boxed{\infty}$

(3) Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$ for all x,

$$\frac{e^x - 1}{x} = \frac{\sum_{n=1}^{\infty} \frac{x^n}{n!}}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \quad \Longrightarrow \quad \int \frac{e^x - 1}{x} dx = \boxed{C + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}}$$

Since integration does not change the radius of convergence, it is $\boxed{\infty}$ and so the interval of convergence is $\boxed{(-\infty, \infty)}$

WebAssign Problem 4 (8pts)

Use power series to evaluate

$$\lim_{x \longrightarrow 0} \frac{1 - \cos x}{1 + x - e^x}$$

Since for x near 0 (in fact, for all x)

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad 1 - \cos x = -\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \sum_{n=2}^{\infty} \frac{x^n}{n!}, \qquad 1 + x - e^x = -\sum_{n=2}^{\infty} \frac{x^n}{n!},$$

we obtain

$$\frac{1 - \cos x}{1 + x - e^x} = \frac{-\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}{-\sum_{n=2}^{\infty} \frac{x^n}{n!}} = \frac{\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}{\sum_{n=2}^{\infty} \frac{x^n}{n!}} = \frac{-\frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots}{\frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots}$$
$$= \frac{-\frac{1}{2!} + \frac{1}{4!}x^2 - \ldots}{\frac{1}{2!} + \frac{1}{3!}x + \ldots} \xrightarrow{x \longrightarrow 0} \frac{-1/2}{1/2} = \boxed{-1}$$

where $\ldots$ on the second line are terms involving positive powers of x, which approach 0 as $x \longrightarrow 0$.

WebAssign Problems 5-9 (4+4+5+5+5pts)

Show that the following series are convergent and find their sums.

$$(5) \ \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} \qquad (6) \ \sum_{n=0}^{\infty} \frac{3^n}{5^n n!} \qquad (7) \ \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!}$$

$$(8) \ \sum_{n=0}^{\infty} \frac{(-1)^n (\ln 2)^n}{n!} \qquad (9) \ \sum_{n=1}^{\infty} \frac{3^n}{n!}$$

(5) First, write this infinite series as some power series evaluated at some point:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\pi}{6}\right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \bigg|_{x=\pi/6}.$$

Since the power series $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$ converges for all x and its sum equals $\cos x$, the evaluation of this power series at $x=\pi/6$, i.e. the infinite series $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{6^{2n}(2n)!}$, also converges and equals

$$\cos\frac{\pi}{6}=\cos 30^\circ=\boxed{\frac{\sqrt{3}}{2}}$$

Remark: You can also justify convergence using one of the convergence tests for infinite series. Since this series involves π^{2n} and $(2n)!$, try the Ratio Test first:

$$\frac{|a_{n+1}|}{|a_n|}=\frac{\pi^{2(n+1)}/(6^{2(n+1)}(2(n+1))!)}{\pi^{2n}/(6^{2n}(2n)!)}=\frac{\pi^{2n+2}}{\pi^{2n}}\cdot\frac{6^{2n}}{6^{2n+2}}\cdot\frac{(2n)!}{(2n+2)!}=\pi^2\frac{1}{6^2}\frac{1}{(2n+1)(2n+2)}\longrightarrow 0.$$

Since $0<1$, the series converges by the Ratio Test. The alternating series test can also be used.

(6) First, write this infinite series as some power series evaluated at some point:

$$\sum_{n=0}^{\infty}\frac{3^n}{5^n n!}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{3}{5}\right)^n=\sum_{n=0}^{\infty}\frac{x^n}{n!}\bigg|_{x=3/5}.$$

Since the power series $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges for all x and its sum equals e^x, the evaluation of this power series at $x=3/2$, i.e. the infinite series $\displaystyle\sum_{n=0}^{\infty}\frac{3^n}{5^n n!}$, also converges and equals $\boxed{\mathrm{e}^{3/5}}$

Remark: You can also justify convergence using one of the convergence tests for infinite series. Since this series involves 3^n and $n!$, try the Ratio Test first:

$$\frac{|a_{n+1}|}{|a_n|}=\frac{3^{n+1}/(5^{n+1}(n+1)!)}{3^n/(5^n n!)}=\frac{3^{n+1}}{3^n}\cdot\frac{5^n}{5^{n+1}}\cdot\frac{n!}{(n+1)!}=3\frac{1}{5}\frac{1}{(n+1)}\longrightarrow 0.$$

Since $0<1$, the series converges by the Ratio Test.

(7) First, write this infinite series as some power series evaluated at some point:

$$\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{4^{2n+1}(2n+1)!}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{4}\right)^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}\bigg|_{x=\pi/4}.$$

Since the power series $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$ converges for all x and its sum equals $\sin x$, the evaluation of this power series at $x=\pi/4$, i.e. the infinite series $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{4^{2n+1}(2n+1)!}$, also converges and equals

$$\sin\frac{\pi}{4}=\boxed{\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}}$$

Remark: You can also justify convergence using one of the convergence tests for infinite series. Since this series involves π^{2n+1} and $(2n+1)!$, try the Ratio Test first:

$$\begin{aligned}\frac{|a_{n+1}|}{|a_n|}=\frac{\pi^{2(n+1)+1}/(4^{2(n+1)+1}(2(n+1)+1)!)}{\pi^{2n+1}/(4^{2n+1}(2n+1)!)}&=\frac{\pi^{2n+3}}{\pi^{2n+1}}\cdot\frac{4^{2n+1}}{4^{2n+3}}\cdot\frac{(2n+1)!}{(2n+3)!}\\&=\pi^2\frac{1}{4^2}\frac{1}{(2n+2)(2n+3)}\longrightarrow 0.\end{aligned}$$

Since $0<1$, the series converges by the Ratio Test. The alternating series test can also be used.

(8) First, write this infinite series as some power series evaluated at some point:

$$\sum_{n=0}^{\infty} \frac{(-1)^n (\ln 2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-\ln 2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}\bigg|_{x=-\ln 2}.$$

Since the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x and its sum equals e^x, the evaluation of this power series at $x=-\ln 2$, i.e. the infinite series $\sum_{n=0}^{\infty} \frac{(-\ln 2)^n}{n!}$, also converges and equals

$$\mathrm{e}^{-\ln 2} = \mathrm{e}^{(-1)\ln 2} = \mathrm{e}^{\ln(2^{-1})} = 2^{-1} = \boxed{\frac{1}{2}}$$

Remark: You can also justify convergence using one of the convergence tests for infinite series. Since this series involves $(\ln 2)^n$ and $n!$, try the Ratio Test first:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(\ln 2)^{n+1}/(n+1)!}{(\ln 2)^n/n!} = \frac{(\ln 2)^{n+1}}{(\ln 2)^n} \cdot \frac{n!}{(n+1)!} = (\ln 2)\frac{1}{(n+1)} \longrightarrow 0.$$

Since $0<1$, the series converges by the Ratio Test. The alternating series test can also be used.

(9) First, write this infinite series as some power series evaluated at some point:

$$\sum_{n=1}^{\infty} \frac{3^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}\bigg|_{x=3} - 1.$$

Since the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x and its sum equals e^x, the evaluation of this power series at $x=3$, i.e. the infinite series $\sum_{n=0}^{\infty} \frac{3^n}{n!}$, also converges and equals e^3; so the original series converges to $\boxed{\mathrm{e}^3-1}$

Remark: You can also justify convergence using one of the convergence tests for infinite series. Since this series involves 3^n and $n!$, try the Ratio Test first:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{3^{n+1}/(n+1)!}{3^n/n!} = \frac{3^{n+1}}{3^n} \cdot \frac{n!}{(n+1)!} = 3\frac{1}{(n+1)} \longrightarrow 0.$$

Since $0<1$, the series converges by the Ratio Test.

Problem XI.1 (5pts)

Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$.

The principle with such problems is to *guess* a function $f(x)$ with a simple power series representation,

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

so that the given power series is obtained by replacing x with some number a. If this a lies in the interval of convergence for the power series, then the sum of the given series is simply $f(a)$. The hard part is usually to guess f correctly.

In the given case, the coefficients in the power series are reciprocals of odd integers $1/(2n+1)$. This is similar to the series in Example 6.10b:

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} \qquad \text{if} \quad -1 < x \leq 1.$$

So, we relate our series to this series:

$$\sum_{n=1}^{\infty}(-1)^n \frac{(1/\sqrt{3})^{2n}}{2n+1} = \sqrt{3}\sum_{n=1}^{\infty}(-1)^n \frac{(1/\sqrt{3})^{2n+1}}{2n+1} = \sqrt{3}\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} - x\right)\Bigg|_{x=1/\sqrt{3}}$$

$$= \sqrt{3}\left(\arctan\left(\frac{1}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}}\right) = \sqrt{3}\left(\frac{\pi}{6} - \frac{1}{\sqrt{3}}\right) = \boxed{-\frac{6-\pi\sqrt{3}}{6}}$$

Problem XI.2 (10pts)

Use power series to estimate $\arctan .2$ *correct within* $\frac{1}{2}\cdot 10^{-5}$. *Leave your answer as a simple fraction* p/q *and determine whether your estimate is an under- or over-estimate.*

By Example 6.10b,

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} \qquad \text{if} \quad -1 < x \leq 1.$$

Since $-1 < .2 \leq 1$, it follows that

$$\arctan .2 = \sum_{n=0}^{\infty}(-1)^n \frac{(1/5)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)5^{2n+1}}.$$

We need to find m so that

$$\left|\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)5^{2n+1}} - \sum_{n=0}^{n=m} \frac{(-1)^n}{(2n+1)5^{2n+1}}\right| < \frac{1}{2}\cdot 10^{-5} = \frac{1}{2\cdot 10^5}.$$

The series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)5^{2n+1}}$ is alternating (odd terms are negative, even terms are positive),

$$\lim_{n\longrightarrow\infty} \frac{1}{(2n+1)5^{2n+1}} = 0, \qquad \frac{1}{(2n+1)5^{2n+1}} > \frac{1}{(2(n+1)+1)5^{2(n+1)+1}}.$$

Thus, the Alternating Series Estimation Theorem (p587) applies and

$$\left|\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)5^{2n+1}} - \sum_{n=0}^{n=m} \frac{(-1)^n}{(2n+1)5^{2n+1}}\right| < |a_{m+1}| = \frac{1}{(2(m+1)+1)5^{2(m+1)+1}}$$

So we need m such that $1/\big((2m+3)5^{2m+3}\big) \leq 1/(2\cdot 10^5)$ or $(2m+3)5^{2m+3} \geq 2\cdot 10^5$. Plugging in small values of m, we find that $m=2$ already works ($m=1$ does not work). So our estimate is

$$\sum_{n=0}^{n=m} \frac{(-1)^n}{(2n+1)5^{2n+1}} = \sum_{n=0}^{n=2} \frac{(-1)^n}{(2n+1)5^{2n+1}} = \frac{(-1)^0}{(2\cdot 0+1)5^{2\cdot 0+1}} + \frac{(-1)^1}{(2\cdot 1+1)5^{2\cdot 1+1}} + \frac{(-1)^2}{(2\cdot 2+1)5^{2\cdot 2+1}}$$

$$= \frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} = \frac{9375-125+3}{3\cdot 15625} = \boxed{\frac{9253}{46875}}$$

Since the last term used is positive, this is an *over*-estimate for arctan .2.

Note: Since arctan $.2 \approx .197396$ and $9253/46875 \approx .197397$, our estimate is indeed within $.5 \cdot 10^{-6}$ and is an *over*-estimate.

Problem XI.3 (10pts)

(a; **4pts**) *By completing the square, show that*

$$\int_0^{1/2} \frac{\mathrm{d}x}{x^2-x+1} = \frac{\pi}{3\sqrt{3}}$$

Since

$$x^2-x+1 = \frac{3}{4}+\big(x-1/2\big)^2 = \frac{3}{4}\left(1+\frac{(x-1/2)^2}{(\sqrt{3}/2)^2}\right) = \frac{3}{4}\left(1+\left(\frac{x-1/2}{\sqrt{3}/2}\right)^2\right) = \frac{3}{4}\Big(1+\big((2x-1)/\sqrt{3}\big)^2\Big),$$

we obtain

$$\begin{aligned}\int_0^{1/2}\frac{\mathrm{d}x}{x^2-x+1} &= \frac{4}{3}\int_0^{1/2}\frac{\mathrm{d}x}{1+((2x-1)/\sqrt{3})^2} = \frac{2}{\sqrt{3}}\int_{-1/\sqrt{3}}^{0}\frac{\mathrm{d}u}{1+u^2} = \frac{2}{\sqrt{3}}\arctan\Big|_{-1/\sqrt{3}}^{0}\\ &= \frac{2}{\sqrt{3}}\big(\arctan 0 - \arctan(-1/\sqrt{3})\big) = \frac{2}{\sqrt{3}}\big(0+\arctan(1/\sqrt{3})\big) = \frac{2}{\sqrt{3}}\cdot\frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}\,,\end{aligned}$$

where $u=(2x-1)/\sqrt{3}$.

(b; **6pts**) *By factoring x^3+1 as a sum of cubes, rewrite the integral in (a). Then express $1/(x^3+1)$ as the sum of a power series and use it to show that*

$$\pi = \frac{3\sqrt{3}}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{8^n}\left(\frac{2}{3n+1}+\frac{1}{3n+2}\right).$$

Since $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ if $|x|<1$,

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty}(-x^3)^n = \sum_{n=0}^{\infty}(-1)^n(x^3)^n = \sum_{n=0}^{\infty}(-1)^n x^{3n}$$

if $|x^3|<1$, or equivalently $|x|<1$. Since $x^3+1^3=(x+1)(x^2-x+1)$,

$$\frac{1}{x^2-x+1} = \frac{1+x}{1+x^3} = (1+x)\sum_{n=0}^{\infty}(-1)^n x^{3n} = \sum_{n=0}^{\infty}(-1)^n\big(x^{3n}+x^{3n+1}\big)$$

if $|x|<1$. Since $|x|<1$ whenever $0<x<1/2$,

$$\begin{aligned}\int_0^{1/2}\frac{\mathrm{d}x}{x^2-x+1} &= \int_0^{1/2}\left(\sum_{n=0}^{\infty}(-1)^n\big(x^{3n}+x^{3n+1}\big)\right)\mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n\left(\frac{x^{3n+1}}{3n+1}+\frac{x^{3n+2}}{3n+2}\right)\Bigg|_0^{1/2}\\ &= \sum_{n=0}^{\infty}(-1)^n\left(\frac{(1/2)^{3n+1}}{3n+1}+\frac{(1/2)^{3n+2}}{3n+2}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{3n}2^2}\left(\frac{2}{3n+1}+\frac{1}{3n+2}\right)\\ &= \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{8^n}\left(\frac{2}{3n+1}+\frac{1}{3n+2}\right)\end{aligned}$$

Comparing this result with the statement in (a), we obtain

$$\pi = \frac{3\sqrt{3}}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{8^n}\left(\frac{2}{3n+1}+\frac{1}{3n+2}\right).$$

Problems XI.4 (10pts)

Find the Taylor series expansion of the function $f(x)=x-x^3$ around $a=-2$ and determine its radius and interval of convergence.

In this case, we can compute all derivatives:

$$\begin{aligned}
&f^{\langle 0\rangle}(-2) = f(-2) = (x-x^3)|_{x=-2} = 6, && f^{\langle 1\rangle}(-2) = f'(-2) = (1-3x^2)|_{x=-2} = -11,\\
&f^{\langle 2\rangle}(-2) = f''(-2) = -6x|_{x=-2} = 12, && f^{\langle 3\rangle}(-2) = f'''(-2) = -6|_{x=-2} = -6,\\
&f^{\langle n\rangle}(-2) = 0 \quad \text{if } n \geq 4.
\end{aligned}$$

Thus, by the main Taylor series formula

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{\langle n\rangle}(-2)}{n!}\big(x-(-2)\big)^n = \boxed{6-11(x+2)+6(x+2)^2-(x+2)^3}$$

Being a finite sum, this series converges for all x (finitely many numbers can always be added together). Thus, the interval of convergence is $\boxed{(-\infty,\infty)}$ and the radius of convergence is $\boxed{\infty}$

Problem J (5pts)

Use Taylor series to obtain Euler's formula:

$$\mathrm{e}^{\mathrm{i}t} = \cos t + \mathrm{i}\sin t.$$

Use the Taylor series expansions at $t=0$ for the exponential, cosine, and sine and $\mathrm{i}^2=-1$:

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}t} = \sum_{n=0}^{\infty}\frac{(\mathrm{i}t)^n}{n!} = \sum_{n=0}^{\infty}\frac{(\mathrm{i}t)^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\frac{(\mathrm{i}t)^{2n+1}}{(2n+1)!} &= \sum_{n=0}^{\infty}\frac{(\mathrm{i}^2)^n t^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\frac{\mathrm{i}(\mathrm{i}^2)^n t^{2n+1}}{(2n+1)!}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!} + \mathrm{i}\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!} = \cos t + \mathrm{i}\sin t.
\end{aligned}$$

Note: Euler's formula is used in solving second-order linear homogeneous differential equations with constant coefficients when the roots of the quadratic polynomial are complex.

Problem K (20pts)

(a; **6pts**) *Let $p(x)$ be any polynomial in x and $n>0$ any positive integer. Show that*

$$\lim_{x\longrightarrow 0} x^{-n}p(x)\mathrm{e}^{-1/x^2} = 0.$$

First, check this for $p(x)=1$:

$$\lim_{x\longrightarrow 0} x^{-n}\mathrm{e}^{-1/x^2} = \lim_{x\longrightarrow 0}\frac{(1/x)^n}{\mathrm{e}^{1/x^2}} = \lim_{x\longrightarrow\infty}\frac{x^n}{\mathrm{e}^{x^2}} = 0;$$

the last equality follows from l'Hospital's rule, since $x^n, \mathrm{e}^{x^2} \longrightarrow \infty$, as do all derivatives of e^{x^2} (each of them is a polynomial multiplied by e^{x^2}). Thus,

$$\lim_{x\longrightarrow 0} x^{-n}p(x)\mathrm{e}^{-1/x^2} = \lim_{x\longrightarrow 0} p(x) \cdot \lim_{x\longrightarrow 0} x^{-n}\mathrm{e}^{-1/x^2} = p(0)\cdot 0 = 0.$$

(b; **12pts**) *Show that the function $f = f(x)$ given by*

$$f(x) = \begin{cases} \mathrm{e}^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0; \end{cases}$$

is smooth and its k-th derivative is a function of the form

$$f^{\langle k\rangle}(x) = \begin{cases} x^{-n_k}p_k(x)\mathrm{e}^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

where n_k is some positive integer and $p_k(x)$ is some polynomial in x.

For $k=0$, $f^{\langle k\rangle}=f$ is indeed of the claimed form, with $n_k=0$ and $p_k(x)=1$. If $f^{\langle k\rangle}$ is of the claimed form for some $k\geq 0$ and $x\neq 0$

$$\begin{aligned} f^{\langle k+1\rangle}(x) &= \big(x^{-n_k}p_k(x)\mathrm{e}^{-1/x^2}\big)' \\ &= -n_k x^{-n_k-1}p_k(x)\mathrm{e}^{-1/x^2} + x^{-n_k}p_k'(x)\mathrm{e}^{-1/x^2} + x^{-n_k}p_k(x)\mathrm{e}^{-1/x^2}(2/x^3) \\ &= x^{-(n_k+3)}\big((2-x^2)p_k(x) + x^3p_k'(x)\big)\mathrm{e}^{-1/x^2}. \end{aligned}$$

For $x=0$, the derivative has to be computed directly from the definition:

$$f^{\langle k+1\rangle}(0) = \lim_{h\longrightarrow 0}\frac{f^{\langle k\rangle}(h)-f^{\langle k\rangle}(0)}{h} = \lim_{h\longrightarrow 0}\frac{h^{-n_k}p_k(h)\mathrm{e}^{-1/h^2}}{h} = \lim_{h\longrightarrow 0} h^{-(n_k+1)}p_k(h)\mathrm{e}^{-1/h^2} = 0;$$

the last equality holds by part (a). Thus, if $f^{\langle k\rangle}$ is of the claimed form for some $k\geq 0$, then $f^{\langle k+1\rangle}$ is of the claimed form with

$$n_{k+1} = n_k + 3, \qquad p_{k+1}(x) = (2-x^2)p_k(x) + x^3p_k'(x).$$

This shows that $f^{\langle k\rangle}$ is of the claimed form for all k. So $f=f(x)$ is a smooth function and $f^{\langle k\rangle}(0)=0$ for all k.

(c; **2pts**) *Conclude that the smooth function $f(x)$ does not admit a Taylor series expansion on any neighborhood of 0 (the Taylor series of f at $x=0$ does not converge to $f(x)$ for any $x\neq 0$).*

By part (b), the Taylor expansion of $f=f(x)$ at $x=0$ would have to be

$$\sum_{n=0}^{\infty}\frac{f^{\langle n\rangle}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{0}{n!}x^n = 0.$$

Since $f(x)>0$ if $x\neq 0$, the Taylor series of f at 0 does not converge to f for any $x\neq 0$.