MAT 127: Calculus C, Spring 2022
Homework Assignment 11

WebAssign Problems due before 9am, Wednesday, 04/27
20% bonus for submissions **before 9am, Saturday, 04/23**

Written Assignment due before 4pm, Wednesday, 04/27
in your instructor's office (L01 in Math 4-101B, L02/3 in Math 3-111)

Please read Sections 6.2.3,6.3, thoroughly before starting on the problem set; looking over Section 8.7 of the WebAssign textbook may be helpful too.

Written Assignment: Problems XI.1-4,J,K (below and next page)
Show your work; correct answers without explanation will receive no credit, unless noted otherwise.

Please write your solutions legibly; the graders will disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first), lecture number (L01, L02, or L03), and HW number in the upper-right corner of the first page.

Problem XI.1

Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)3^n}$.

Problem XI.2

Use power series to estimate arctan .2 correct within $\frac{1}{2}\cdot 10^{-5}$. Leave your answer as a simple fraction p/q and determine whether your estimate is an *under*- or *over*-estimate.

Problem XI.3

(a; **4pts**) By completing the square, show that

$$\int_0^{1/2}\frac{\mathrm{d}x}{x^2-x+1}=\frac{\pi}{3\sqrt{3}}$$

(b) By factoring x^3+1 as a sum of cubes, rewrite the integral in (a). Then express $1/(x^3+1)$ as the sum of a power series and use it to show that

$$\pi=\frac{3\sqrt{3}}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{8^n}\left(\frac{2}{3n+1}+\frac{1}{3n+2}\right).$$

Problems XI.4

Find the Taylor series expansion of the function $f(x)=x-x^3$ around $a=-2$ and determine its radius and interval of convergence.

Problem J

Use Taylor series to obtain *Euler's formula*:

$$\mathrm{e}^{\mathrm{i}t} = \cos t + \mathrm{i}\sin t.$$

This is used to solve second-order linear homogeneous ODEs with constant coefficients.
Hint: expand LHS into a power series and separate real and imaginary terms.

Problem K

(a) Let $p(x)$ be any polynomial in x and $n > 0$ any positive integer. Show that

$$\lim_{x \longrightarrow 0} x^{-n} p(x) \mathrm{e}^{-1/x^2} = 0.$$

Hint: first do this for $p(x)=1$; replacing x by $1/x$ may simplify l'Hospital.

(b) Show that the function $f = f(x)$ given by

$$f(x) = \begin{cases} \mathrm{e}^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0; \end{cases}$$

is smooth and its k-th derivative is a function of the form

$$f^{\langle k \rangle}(x) = \begin{cases} x^{-n_k} p_k(x) \mathrm{e}^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

where n_k is some positive integer and $p_k(x)$ is some polynomial in x.
Hint: This function is obviously smooth outside of $x=0$. Use induction on k to show that its k-th derivative has the above form outside of $x=0$. Use the limit definition of derivative to show that each of the above functions is differentiable at $x=0$.

(c) Conclude that the smooth function $f(x)$ does not admit a Taylor series expansion on any neighborhood of 0 (the Taylor series of f at $x=0$ does not converge to $f(x)$ for any $x \neq 0$).