Please read Section 8.7 thoroughly before starting on the problem set.

**Written Assignment:** 8.7 12 (also determine radius/interval of convergence); Problems H,I (below)

*Please write your solutions legibly; the graders may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name and lecture number in the upper-right corner of the first page.*

**Problem H**

Use Taylor series to obtain *Euler’s formula*:

\[ e^{it} = \cos t + i \sin t. \]

This is used to solve second-order linear homogeneous ODEs with constant coefficients.

*Hint:* expand LHS into a power series and separate real and imaginary terms.

**Problem I**

(a) Let \( p(x) \) be any polynomial in \( x \) and \( n > 0 \) any positive integer. Show that

\[ \lim_{x \to 0} x^{-n}p(x)e^{-1/x^2} = 0. \]

*Hint:* first do this for \( p(x) = 1 \); replacing \( x \) by \( 1/x \) may simplify l’Hospital.

(b) Show that the function \( f = f(x) \) given by

\[ f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0; \end{cases} \]

is smooth and its \( k \)-th derivative is a function of the form

\[ f^{(k)}(x) = \begin{cases} x^{-nk}p_k(x)e^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} \]

where \( n_k \) is some positive integer and \( p_k(x) \) is some polynomial in \( x \).

*Hint:* This function is obviously smooth outside of \( x = 0 \). Use induction on \( k \) to show that its \( k \)-th derivative has the above form outside of \( x = 0 \). Use the limit definition of derivative to show that each of the above functions is differentiable at \( x = 0 \).

(c) Conclude that the smooth function \( f(x) \) does not admit a Taylor series expansion on any neighborhood of 0 (the Taylor series of \( f \) at \( x = 0 \) does not converge to \( f(x) \) for any \( x \neq 0 \)).