Problem Set 5: Notes 1, 7, 12, 16; Problem D (below)

Show your work; correct answers without explanation will receive no credit, unless noted otherwise.

Please write your solutions legibly; the graders may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name and lecture number in the upper-right corner of the first page.

Problem D

By part (c) of Problem B on HW2, the first-order differential equation
\[ y' - by = f(x), \quad y = y(x), \quad b = \text{const}, \]
can be solved by multiplying both sides by \( e^{-bx} \). This equation then becomes
\[ (e^{-bx} y)' = e^{-bx} f(x) \]
and can be solved by integrating both sides. Note that \( b \) is the root of the associated linear equation \( r - b = 0 \). This approach has an analogue for second-order inhomogeneous linear equations
\[ y'' + by' + cy = f(x), \quad y = y(x), \quad b, c = \text{const}. \quad (1) \]

(a) If \( r_1, r_2 \) are the two roots of the quadratic equation \( r^2 + br + c = 0 \) associated to (1), show that
\[ \left( e^{(r_1-r_2)x} (e^{-r_1x} y) \right)' = e^{-r_2x} (y'' + by' + cy). \quad \quad (2) \]

By (2), equation (1) is equivalent to
\[ \left( e^{(r_1-r_2)x} (e^{-r_1x} y) \right)' = e^{-r_2x} f(x), \quad y = y(x), \quad (3) \]
which can be solved by integrating twice.

(b,c) Find the general solutions \( y = y(x) \) of the differential equations

(b) \( y'' + 5y' + 4y = e^{-x} \), \hspace{1cm} (c) \( y'' + 4y = 4 \cos 2x \).

Hint: In both cases, choose the order of the two roots wisely to get the simpler of the two possible versions of RHS in (3). In (c), it is simpler to replace \( \cos 2x \) by \( e^{2ix} \) and then take the real part of the general solution, which will be the general (real) solution to (c) because \( \cos 2x \) is the real part of \( e^{2ix} \) and all coefficients in the equation are real.