MATH 127 Solutions to First Midterm

1. A culture of bacteria grows at a rate proportional to the number of bacteria present in the culture. At noon on January 24, there were 5 thousand bacteria. At 2 PM, there were 20 thousand present.

(a) 12 points Give a formula for $B(t)$, the number of bacteria in the culture $t$ hours after noon on January 24.

**Solution:** Since the growth rate of the bacteria is proportional to the number present, we have the differential equation

$$B'(t) = kB(t)$$

where $t$ is the time in hours since noon, and $B(t)$ is the number of bacteria in thousands.

This is a separable equation, so we separate variables to obtain

$$\int \frac{dB}{B} = \int k \, dt$$

$$\ln |B| = kt + c$$

exponentiating both sides,

$$|B| = e^{kt+c}$$

so, since we can write $\pm e^c$ as an arbitrary constant $A$, we have

$$B = Ae^{kt}.$$ 

(Many students just remembered the formula for exponential growth and skipped directly to this step. That’s fine, too.)

From the initial condition, we know $B(0) = 5 = Ae^0 = A$. Since we also have $B(2) = 20$, we can solve for $k$:

$$20 = 5e^{2k}$$

$$4 = e^{2k}$$

Taking logs,

$$\ln 4 = 2k$$

so $k = \frac{\ln 4}{2}$, or $k = \ln 2$.

So

$$B(t) = 5e^{t\ln 2} = 5 \cdot 2^t.$$ 

(either form is OK. Many people also wrote $5e^{\frac{\ln 4}{2}t}$, which is equivalent.)
(b) **8 points** When will there be 100 thousand bacteria in the culture?

**Solution:** To answer this, we need to find the value of $t$ so that $B(t) = 100$. Since we have $B(t) = 5e^{t\ln 2}$ from the previous part, we solve

$$100 = 5e^{t\ln 2}$$

$$20 = e^{t\ln 2}$$

Now take the log of both sides,

$$\ln 20 = t \ln 2$$

$$\frac{\ln 20}{\ln 2} = t$$

That is, about 4.3 hours after noon.

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2. **20 points** Consider the initial value problem given by

$$y' = 2x - y \quad y(0) = 0$$

Use Euler’s method with a stepsize $h = 1$ to find an approximation to $y(3)$. To receive full credit, show your intermediate steps *clearly*.

**Solution:** Our initial point on our numeric solution is $(x_0, y_0) = (0, 0)$. The next approximation is given by $x_1 = x_0 + h$ and $y_1 = y_0 + h \cdot y'(x_0, y_0)$, so we need to find the slope of the solution at $(0, 0)$. Since our stepsize $h = 1$, things are easier.

$$y'(0, 0) = 2 \cdot 0 - 0 = 0 \quad \text{so} \quad (x_1, y_1) = (1, 0 + 0) = (1, 0).$$

Now we compute the slope at $(1, 0)$ for the next point. We have

$$y'(1, 0) = 2 \cdot 1 - 0 = 2 \quad \text{so} \quad (x_2, y_2) = (2, 0 + 2) = (2, 2).$$

Continuing in this way,

$$y'(2, 2) = 2 \cdot 2 - 2 = 2 \quad \text{so} \quad (x_3, y_3) = (3, 2 + 2) = (3, 4).$$

Our final approximation is then $y(3) = 4$. 

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3. Consider the second order linear differential equation

\[ y'' - 4y = 0 \]

(a) **10 points** Write a formula for the general solution \( y(t) \).

**Solution:** We look for solutions of the form \( y = e^{kt} \), so we plug this in to get

\[ k^2 e^{kt} - 9e^{kt} = 0. \]

This factors as

\[ e^{kt}(k - 3)(k + 3) = 0, \]

which only has solutions when \( k = 3 \) or \( k = -3 \). This means the general solution to this differential equation is

\[ y = Ae^{3t} + Be^{-3t}, \]

where \( A \) and \( B \) are arbitrary constants.

(b) **10 points** Let \( y(t) \) be the specific solution with \( y(0) = 1 \) and \( y'(0) = 0 \). Write a formula for \( y(t) \).

**Solution:** We need to determine \( A \) and \( B \) subject to the given initial conditions. From \( y(0) = 1 \), we have

\[ 1 = Ae^0 + Be^0 = A + B \quad \text{so} \quad B = 1 - A. \]

That is, \( y(t) = Ae^{3t} + (1 - A)e^{-3t} \), and so

\[ y'(t) = 3Ae^{3t} - 3(1 - A)e^{-3t}. \]

Plugging in \( y'(0) = 0 \) gives us

\[ 0 = 3A - 3(1 - A) \quad \text{that is} \quad 0 = 6A - 3 \quad \text{or} \quad A = \frac{1}{2} \]

Hence, \( B = 1/2 \) and our solution is

\[ y(t) = \frac{e^{3t}}{2} + \frac{e^{-3t}}{2}. \]
4. The direction field for a differential equation is shown below.

(a) [15 points] On the direction field, sketch and clearly label the three solutions with initial conditions

\[ y_1(0) = 0.5 \quad y_2(0) = 1 \quad y_3(0) = 2 \]

(b) [5 points] Are there any equilibrium solutions (also called stationary solutions, or constant solutions)? If there are, identify them. If not, give a reason why not.

Solution: There are no equilibrium solutions (at least not for \(0 \leq y \leq 2\)). If there were, such a solution would be of the form \(y(x) = c\) for some constant \(c\), and its graph would be a horizontal line. Along this solution, the direction field must be slope 0 for all \(x\). Since there are no such lines in the given direction field, we can have no equilibrium solutions.
5. Write solutions to the following initial-value problems.

(a) \[ 10 \text{ points} \]
\[ y' = \frac{e^{3x}}{y^2} \quad y(0) = 2 \]

**Solution:** This is a separable equation, so we separate the variables to obtain
\[ \int y^2 \, dy = \int e^{3x} \, dx \]
and so
\[ \frac{y^3}{3} = \frac{e^{3x}}{3} + c \]
\[ y = \sqrt[3]{e^{3x} + c} \]

Now using the initial condition \( y(0) = 2 \), we have
\[ 2 = \sqrt[3]{1 + c} \]
So \( c = 7 \) and our solution is
\[ y = \sqrt[3]{e^{3x} + 7} \]

(b) \[ 10 \text{ points} \]
\[ y' = 1 + y^2 \quad y(1) = 0 \]

**Solution:** This equation is also separable. Separating gives
\[ \int \frac{dy}{1 + y^2} = \int dx \]
so
\[ \arctan y = x + c \quad \text{and hence} \quad y = \tan(x + c) \]

The initial condition gives us \( 0 = \tan(1 + c) \), and since \( \tan 0 = 0 \), we know that \( c = -1 \). Hence the desired solution is
\[ y = \tan(x - 1) \]