Quiz 2.

Problem 1. Solve the initial value problem
\[ y' = \frac{3y + x}{x}, \quad y(1) = \frac{3}{2}. \]

Problem 2. Find the general solution of the differential equation
\[ yy'' = (y')^2. \]
Problem 1  First order linear ODE

\[ y' = 3 \frac{y}{x} + 1 \]
\[ y' - \frac{3}{x} y = 1 \]

Integrating factor \( \rho(x) = e^{\int -\frac{3}{x} \, dx} = e^{-3 \ln x} = x^{-3} \)

\[ x^{-3} y' - \frac{3}{x} x^{-3} y = x^{-3} \]
\[ \frac{d}{dx} (x^{-3} y) = x^{-3} \]
\[ x^{-3} y = \int \frac{d}{dx} (x^{-3} y) \, dx = \int x^{-3} \, dx = -\frac{1}{2} x^{-2} + C \]

so \( y(x) = -\frac{1}{2} x + C x^3 \)

Initial condition:
\[ \frac{3}{2} = y(1) = -\frac{1}{2} + C \quad \Rightarrow \quad C = 2 \]

so \( y(x) = -\frac{1}{2} x + 2 x^3 \)

Alternative solution: homogeneous equation

\[ \frac{dy}{dx} = 3 \left( \frac{y}{x} \right) + 1 = F \left( \frac{y}{x} \right) \]
Problem 2  Reducible second order ODE

\[ y \frac{d^2 y}{dx^2} = \left( \frac{dy}{dx} \right)^2 \]  \hspace{1cm} (1)

We use substitution \( u = \frac{dy}{dx} \) and find a differential equation for \( u \) as a function of \( y \). Because \( u = \frac{dy}{dx} \),

\[ \frac{d^2 y}{dx^2} = \frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx} = \frac{du}{dy} u \]

\[ \text{chain rule} \]

so equation (1) is equivalent to

\[ y \frac{du}{dy} u = u^2 \]

Assume \( u \neq 0 \) (if \( u = 0 \), then \( y = \text{const.} \)) and divide by \( u \)

\[ y \frac{du}{dy} = u \]

separable equation

\[ \int \frac{du}{u} = \int \frac{dy}{y} \implies \ln u = \ln y + C \]

\[ \implies u = e^C y = Ay \]

Now, \( \frac{dy}{dx} = u = Ay \)

We solve for \( y = y(x) \) by separating variables

\[ \int \frac{dy}{Ay} = \int dx = Ax + C \]

\[ \ln y = Ax + C \implies y(x) = e^{Ax+C} = e^C e^{Ax} = Be^{Ax} \]

\[ A, B = \text{any constants} \]