Problem 1. A car traveling at 30 \( mi/h \) (44 \( ft/s \)) gradually speeds up during 10 seconds with the acceleration given by
\[
a(t) = 0.06t^2 + 2.4 \ (ft/s^2).
\]
Find the distance it has traveled in these 10 seconds and its velocity at the end.

Solution. Since acceleration is the derivative of the speed (for movement by a straight line), we obtain the following differential equation:
\[
\frac{dv}{dt} = 0.06t^2 + 2.4.
\]
Solving it by integration, we find:
\[
v(t) = \int (0.06t^2 + 2.4)dt = 0.02t^3 + 2.4t + v_0 = 0.02t^3 + 2.4t + 44 \ (ft/s).
\]
In particular, the velocity after 10s is
\[
v(10) = 0.02 \cdot 10^3 + 2.4 \cdot 10 + 44 = 88 \ (ft/s).
\]
Since velocity is the derivative of the distance (for movement by a straight line), we have:
\[
\frac{dx}{dt} = v(t) = 0.02t^3 + 2.4t + 44.
\]
Thus,
\[
x(t) = \int (0.02t^3 + 2.4t + 44)dt = 0.005t^4 + 1.2t^2 + 44t.
\]
In particular, after 10s the car traveled
\[
0.005 \cdot 10^4 + 1.2 \cdot 10^2 + 44 \cdot 10 = 610 \ (ft).
\]
Answer: The velocity at the end is 88 \( ft/s \), the distance traveled is 610 \( ft \).
Problem 2. Solve the initial value problem

\[ x \frac{dy}{dx} = y + x^2, \quad y(1) = 0. \]

Solution. This is a linear differential equation. First, rewrite it in the standard form:

\[ \frac{dy}{dx} - \frac{y}{x} = x. \]

Calculate the multiplier:

\[
\rho(x) = \exp\left(\int \left(-\frac{1}{x}\right) dx\right) = \exp(-\ln|x|) = \frac{1}{|x|}.
\]

Since the initial condition is given at the point \( x_0 = 1 > 0 \), we can restrict our attention to \( x > 0 \), so that \( \rho(x) = \frac{1}{x} \). Multiplying the differential equation by \( \rho(x) \), we obtain:

\[
\frac{d}{dx}(\frac{y}{x}) = \frac{1}{x} \left( \frac{dy}{dx} - \frac{1}{x} y \right) = 1.
\]

Therefore,

\[
\frac{y}{x} = \int 1 dx = x + C, \quad y = x^2 + Cx.
\]

Plugging the initial condition \( y(1) = 0 \), we get:

\[ 0 = y(1) = 1 + C. \]

Thus, \( C = -1 \).

Answer: \( y(x) = x^2 - x \).
Problem 3. Solve the differential equation

\[ y' = (2x - y)^2 + 3. \]

Solution. On the right hand side of the equation we see a noticeable expression \(2x - y\), therefore it is natural to try the substitution \(v = 2x - y\). We have: \(v' = 2 - y'\). Thus,

\[ v' = 2 - ((2x - y)^2 + 3) = -1 - v^2. \]

This is a separable equation. Separating variables and integrating, we get

\[ \frac{dv}{v^2 + 1} = -dx, \quad \tan^{-1}(v) = -x + C, \quad v = \tan(C - x). \]

Thus, we obtain:

\[ y = 2x - v = 2x - \tan(C - t) = 2x + \tan(x - C). \]

Since \(C\) is arbitrary constant, we can replace \(C\) with \(-C\).

Answer: \(y(x) = 2x + \tan(x + C)\), where \(C\) is an arbitrary constant.
Problem 4. Show that the following differential equation is exact; then solve it.

\[ \frac{1}{yx} \, dy + (2x - \frac{\ln y}{x^2}) \, dx = 0. \]

Solution. Let \( M(x, y) = \frac{1}{yx} \), \( N(x, y) = 2x - \frac{\ln y}{x^2} \). By Criteria of Exactness, the differential equation \( Mdy + Ndx = 0 \) is exact if and only if \( \frac{\partial M}{\partial x} = \frac{\partial N}{\partial y} \).

We have:

\[ \frac{\partial M}{\partial x} = -\frac{1}{yx^2} = \frac{\partial N}{\partial y}, \]

therefore the equation is exact.

To solve the equation we need to find a function \( F(x, y) \) such that \( \frac{\partial F}{\partial y} = M \), \( \frac{\partial F}{\partial x} = N \). Integrating the first equation, we obtain:

\[ F = \int M \, dy = \int \frac{1}{yx} \, dy = \frac{\ln y}{x} + g, \]

where \( g = g(x) \) is a function of \( x \). Substituting this formula into the second equation, we find:

\[ 2x - \frac{\ln y}{x^2} = \frac{\partial F}{\partial x} = -\frac{\ln y}{x^2} + g'(x). \]

Therefore, \( g'(x) = 2x \), \( g(x) = x^2 + C \). Since we need to find a function \( F(x, y) \), we can fix any \( C \). Let \( C = 0 \). Then \( F(x, y) = \frac{\ln y}{x} + x^2 \).

The general solution of the exact equation is \( F(x, y) = C \), where \( C \) is any constant. Thus,

\[ \frac{\ln y}{x} + x^2 = C, \ln y = Cx - x^3, \ y = \exp(Cx - x^3). \]

Answer: \( y(x) = \exp(Cx - x^3) \), where \( C \) is a constant.
Problem 5. Assume that a population of size $P(t)$ has a constant birth rate $\beta = 0.1$ and the death rate is given by $\delta = 0.01P$.

1) Write the logistic equation which is satisfied by this population.

2) Write the general solution of this logistic equation.

3) Indicate the critical points and their types (stable or unstable).

Solution.

1) The differential equation of the population with given birth and death rates is:

$$\frac{dP}{dt} = (\beta - \delta)P = (0.1 - 0.01P)P = 0.01P(10 - P).$$

2) The general solution of a logistic equation $\frac{dP}{dt} = kP(M - P)$ is:

$$P(t) = \frac{MP_0}{P_0 + (M - P_0) \exp(-kMt)}.$$ 

In our case, $k = 0.01, \ M = 10$. Thus,

$$P(t) = \frac{10P_0}{P_0 + (1 - P_0) \exp(-0.1t)}.$$ 

3) For a logistic equation there are two critical points: 0 and $M$. The critical point 0 is unstable, the critical point $M = 10$ is stable.