MAT 303 Assignment 2 solutions.

**Problem 1.** Consider the differential equation

\[ y' = (x + y)^{1/3} - 1. \tag{1} \]

Describe all pairs of numbers \((x_0, y_0)\) for which Theorem of Existence and Uniqueness guaranties that the initial value problem \(y(x_0) = y_0\) has a unique solution.

**Solution.** Theorem of Existence states the following. If \(f(x, y)\) is continuous in some rectangle containing the point \((x_0, y_0)\), then the initial value problem

\[ y' = f(x, y), \quad y(x_0) = y_0 \]

has a solution (not necessarily unique) on some interval containing \(x_0\). Since \(f(x, y) = (x + y)^{1/3} - 1\) is continuous everywhere, Theorem of Existence guaranties existence of a solution of (1) for all \((x_0, y_0)\).

Theorem of Uniqueness states the following. If \(f(x, y)\) and \(\frac{df}{dy} f(x, y)\) both are continuous in some rectangle containing point \((x_0, y_0)\), then the initial value problem

\[ y' = f(x, y), \quad y(x_0) = y_0 \]

has a unique solution on some interval containing \(x_0\). The functions

\[ f(x, y) = (x + y)^{1/3} - 1, \quad \text{and} \quad \frac{df}{dy} f(x, y) = \frac{1}{3} (x + y)^{-2/3} \]

are continuous at all points except where \(x = -y\). Therefore, Theorem of Uniqueness guaranties uniqueness of solution for all
Problem 2. Solve the differential equation (1). Describe all pairs \((x_0, y_0)\) for which the initial value problem \(y(x_0) = y_0\):
   - \((a)\) has a unique solution,
   - \((b)\) do not have any solutions,
   - \((c)\) has more than one solution.

Solution. Substitute \(v = x + y\). We have: \(\frac{dv}{dx} = \frac{dy}{dx} + 1\). Therefore,
\[
\frac{dv}{dx} = v^{\frac{1}{3}} - 1 + 1 = v^{\frac{1}{3}}.
\]
This is a separable equation. Assuming \(v \neq 0\), we obtain:
\[
v^{-\frac{1}{3}} dv = dx, \quad \int v^{-\frac{1}{3}} dv = \int dx,
\]
\[
\frac{3}{2} v^{\frac{2}{3}} = x + C \Rightarrow v = \pm \left(\frac{2}{3} x + \frac{2}{3} C\right)^{\frac{3}{2}}.
\]
Thus,
\[
y = \pm \left(\frac{2}{3} x + \frac{2}{3} C\right)^{\frac{3}{2}} - x.
\]
Since we assumed \(v \neq 0\), we need to check whether \(v = 0\) is a solution. Clearly, it is a solution. Therefore, \(y = -x\) is a particular solution of (1). Thus, the solutions of (1) are
\[
y = \pm \left(\frac{2}{3} x + C\right)^{\frac{3}{2}} - x, \text{ where } C \text{ is a constant}; \quad y = -x. \quad (2)
\]
Let \((x_0, y_0)\) be a pair of numbers. Let us find all solutions of (1) such that \(y(x_0) = y_0\). Observe that the solution \(y = -x\) satisfies the initial condition \(y(x_0) = y_0\) if and only if \(y_0 = -x_0\). The solution \(y = \pm \left(\frac{2}{3} x + C\right)^{\frac{3}{2}} - x\) satisfies the initial condition \(y(x_0) = y_0\) if and only if
\[
y_0 = \pm \left(\frac{2}{3} x_0 + C\right)^{\frac{3}{2}} - x_0.
\]
This condition defines $C$ uniquely, namely:

$$C = (y_0 + x_0)^\frac{2}{3} - \frac{2}{3}x_0.$$ 

Thus, if $y_0 \neq -x_0$, then the initial value problem has a unique solution and it is of the form

$$y = \left(\frac{2}{3}x + C\right)^\frac{3}{2} - x$$

for some $C$. If $y_0 = -x_0$ then the initial value problem has 2 solutions: $y = -x$ and a solution of the form (3). The answer for the second part of the problem is:

a) all pairs such that $y_0 \neq -x_0$,

b) no such pairs;

c) all pairs of the form $(x_0, -x_0)$.

**Problem 3.** Separate variables and use partial fractions to solve the initial value problem

$$\frac{dx}{dt} = 3x(5 - x), \quad x(0) = 8.$$ 

**Solution.** Observe that $x = 0$ and $x = 5$ are particular solutions of the equation $\frac{dx}{dt} = 3x(5 - x)$. If $x \neq 0$ and $x \neq 5$, we have:

$$\frac{dx}{3x(5-x)} = dt.$$ 

Use partial fractions:

$$\frac{1}{x(5-x)} = \frac{A}{x} + \frac{B}{5-x} = \frac{A(5-x)+Bx}{x(5-x)}.$$ 

Then $5A + (B - A)x = 1$. Therefore, $A = \frac{1}{5}$ and $B = A$. Thus,

$$\int \frac{dx}{3x(5-x)} = \frac{1}{15} \int \left(\frac{1}{x} + \frac{1}{5-x}\right)dx = \frac{1}{15} (\ln |x| - \ln |5 - x|) = \frac{1}{15} \ln \left|\frac{x}{5-x}\right| = t + C.$$
Exponentiating, we obtain:

\[
\frac{x}{5-x} = Ke^{15t}, \quad \text{where } K = \text{const.}
\]

It is convenient to plug the initial condition \(x(0) = 8\) in this formula. We obtain:

\[-\frac{8}{3} = K.\]

Thus,

\[
x(t) = \frac{5Ke^{15t}}{1+Ke^{15t}} = \frac{40e^{15t}}{8e^{15t} - 3}.
\]

**Problem 4.** A tank contains 1000 liters (\(L\)) of a solution consisting of 100 kg of salt dissolved in water. Pure water is pumped into the tank at the rate of 5\(L/s\), and the mixture – kept uniform by stirring – is pumped out at the same rate. How long will it be until only 10 kg of salt remains in the tank?

**Solution.** Observe that no new salt is coming into the tank, but some salt is leaving the tank with the solution. Thus, the amount of salt in the tank changes with time. Let \(x(t)\) be the amount of salt in the tank at time \(t\). Then the ratio of the salt in the solution is \(x/1000\) (\(kg/L\)). Therefore, salt leaves the tank with the speed

\[
5 \cdot \frac{x}{1000} = \frac{x}{200} \text{ (kg/s)}.
\]

We obtain that

\[
\frac{dx}{dt} = -\frac{x}{200}.
\]

Solving this separable equation we find

\[
\frac{dx}{x} = -\frac{dt}{200}, \quad \ln |x| = C - \frac{t}{200},
\]

\[
x = Ae^{-\frac{t}{200}}, \quad \text{where } A = \text{const.}
\]
Observe that $x(0) = 100$ kg (the amount of salt at the beginning). Therefore, $A = 100$. Thus, $x(t) = 100e^{-\frac{t}{200}}$. Solving the equation

$$100e^{-\frac{t}{200}} = 10$$

we find that 10 kg of salt remains after $t = -200 \ln \frac{1}{10} \approx 461$ seconds.

**Problem 5.** Verify that the given differential equation is exact; then solve it.

$$\frac{1}{x} \sin y \, dx + (\ln x \cos y + y) \, dy = 0.$$

**Solution.** By Criteria of Exactness, a differential equation $M(x,y)dx + N(x,y)dy$ is exact if and only if $\frac{dM}{dy} = \frac{dN}{dx}$. We have:

$$\frac{d}{dy} \left(\frac{1}{x} \sin y\right) = \frac{1}{x} \cos y,$$

$$\frac{d}{dx} \left(\ln x \cos y + y\right) = \frac{1}{x} \cos y.$$

Therefore, the equation is exact. To solve it, we need to find $F(x,y)$ such that

$$\frac{dF}{dx} = \frac{1}{x} \sin y,$$

$$\frac{dF}{dy} = \ln x \cos y + y.$$

Then the solution would be $F(x,y) = C$, where $C$ is arbitrary constant.

From the formula for $\frac{dF}{dx}$ we obtain:

$$F = \int \frac{1}{x} \sin y \, dx = \ln x \sin y + g(y).$$

Plugging this formula into the formula for $\frac{dF}{dy}$ we obtain:

$$\ln x \cos y + y = \frac{dF}{dy} = \frac{d}{dy} \left(\ln x \sin y + g(y)\right) = \ln x \cos y + g'(y).$$
Thus, \( g'(y) = y, \) \( g(y) = \frac{y^2}{2} + \text{const.} \) For simplicity, take \( \text{const} = 0. \) Then

\[
F(x, y) = \ln x \sin y + \frac{y^2}{2}.
\]

Thus, the solution of the exact equation is:

\[
\ln x \sin y + \frac{y^2}{2} = C.
\]

**Problem 6.** Show that the following differential equation is homogeneous:

\[
x(\ln x - \ln t + 1)\,dt = t\,dx.
\]

Solve the initial value problem \( x(1) = 1. \)

**Solution.** A first-order differential equation is called homogeneous if it can be written in the form

\[
\frac{dx}{dt} = F(x/t).
\]

We have:

\[
\frac{dx}{dt} = \frac{x}{t}(\ln x - \ln t + 1) = \frac{x}{t}(\ln \frac{x}{t} + 1).
\]

Therefore, this differential equation is homogeneous. To solve it, use the substitution \( v = \frac{x}{t}. \) One has:

\[
x = vt \Rightarrow \frac{dx}{dt} = v + t\frac{dv}{dt}.
\]

We obtain:

\[
v + t\frac{dv}{dt} = v(\ln v + 1), \quad \frac{dv}{v\ln v} = \frac{dt}{t},
\]

\[
\int \frac{dv}{v\ln v} = \int \frac{dt}{t}, \quad \ln |\ln v| = \ln |t| + C.
\]

Thus,

\[
\ln v = Kt, \quad \text{or} \quad v = e^{Kt},
\]
where $K$ is some constant. Finally,

$$x(t) = vt = te^{Kt}.$$  

Substituting the initial condition $x(1) = 1$ we obtain: $1 = 1e^K \Rightarrow K = 0$. Thus, the solution of the initial value problem is $x(t) = t$.

**Problem 7.** The time rate of change of a rabbit population $P$ is proportional to the square root of $P$. At time $t = 0$ (months) the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be one year later?

**Solution.** This rabbit population $P(t)$ satisfies the differential equation

$$\frac{dP}{dt} = k\sqrt{P},$$  \hspace{1cm} (4)

where $k$ is some constant. According to the conditions of the problem, we have: $P(0) = 100$, $\frac{dP}{dt} = 20$. Thus, plugging $t = 0$ into the equation (4) we obtain:

$$20 = k\sqrt{100} \Rightarrow k = 2.$$  

Solving the differential equation $\frac{dP}{dt} = 2\sqrt{P}$ we find:

$$\frac{dP}{2\sqrt{P}} = dt, \hspace{0.5cm} \sqrt{P} = t + C.$$  

For $t = 0$ we obtain: $\sqrt{100} = 0 + C \Rightarrow C = 10$. Thus,

$$P(t) = (10 + t)^2.$$  

So, after 1 year (12 months) there will be $P(12) = (10 + 12)^2 = 484$ rabbits.